## Two complete examples of quotient groups

**Example 1:** Let $G = S_4$, a symmetric group with 24 elements. Let

$$H = \{e, (12)(34), (13)(24), (14)(23)\} \leq G$$

Notice that $H$ is isomorphic to the Klein 4-group. We can calculate the left and right cosets:

$$\begin{aligned}
H &= \{e, (12)(34), (13)(24), (14)(23)\} \\
(12)H &= \{(12), (34), (1324), (1423)\} \\
(13)H &= \{(13), (1234), (24), (1432)\} \\
(14)H &= \{(14), (1243), (1342), (23)\} \\
(123)H &= \{(123), (134), (243), (142)\} \\
(132)H &= \{(132), (234), (124), (143)\}
\end{aligned}$$

$$\begin{aligned}
H &= \{e, (12)(34), (13)(24), (14)(23)\} \\
H(12) &= \{(12), (34), (1423), (1324)\} \\
H(13) &= \{(13), (1432), (24), (1234)\} \\
H(14) &= \{(14), (1342), (1243), (23)\} \\
H(123) &= \{(123), (243), (142), (134)\} \\
H(132) &= \{(132), (143), (234), (124)\}
\end{aligned}$$

Notice that $gH = Hg$ for all $g \in S_n$, but it is not true that $gh = hg \forall h \in H$. Thus $H \trianglelefteq G$, so we can construct a quotient group.

**Operation in** $G/H$: Suppose we want to compute $[(123)H][(14)]H$. We just use the rule:

$$\begin{aligned}
[(123)H][(14)H] &= (123)(14)H \\
&= (1423)H \\
&= (12)H
\end{aligned}$$

where the last equation comes from looking up in the list of cosets.

We can continue this way to get the entire Cayley table for $S_4/H$:

| $S_4/H$ | H | (12)H | (13)H | (14)H | (123)H | (132)H |
|---|---|---|---|---|---|---|
| H | H | (12)H | (13)H | (14)H | (123)H | (132)H |
| (12)H | (12)H | H | (132)H | (123)H | (14)H | (13)H |
| (13)H | (13)H | (123)H | H | (132)H | (12)H | (14)H |
| (14)H | (14)H | (132)H | (123)H | H | (13)H | (12)H |
| (123)H | (123)H | (13)H | (14)H | (12)H | (132)H | H |
| (132)H | (132)H | (14)H | (12)H | (13)H | H | (123)H |

**Remarks:**
1. We chose $\{e, (12), (13), (14), (123), (132)\}$ as our system of coset representatives, but a different choice would give the same quotient group, just with different *labels* of the elements.

2. The quotient group is nonabelian, and is actually isomorphic to $S_3$, we write:

$$S_4/H \cong S_3.$$

**Example 2:** Let $G = GL_n(R)$, the invertible $n \times n$ matrices with real entries.

$$H = SL_n(R) = \{A \in GL_n(R) \mid \det(A) = 1.$$

We know $H \leq G$. Now suppose $g \in H$ and $g \in G$. Then $\det(h) = 1$ and $\det(ghg^{-1}) = det(h) = 1$ so $ghg^{-1} \in H$, i.e. $H \trianglelefteq G$.

This example is different than the previous one since $G$ is infinite, so we may not be able to just write down all the cosets. We can still try to get a system of representatives though. Recall that $aH = bH$ if and only if $b^{-1}a \in H$. But $b^{-1}a \in H$ if and only if $\det(b^{-1}a) = 1$ if and only if $\det(a) = \det(b)$. Thus:

**Key fact:** Two matrices are in the same coset of $H$ if and only if they have the same determinant.

Thus we can think of the cosets as being labelled by nonzero real numbers. Also $aHbH = abH$ and $\det(ab) = \det(a)\det(b)$ so the quotient group $G/H$ is just the nonzero real numbers under multiplication:

$$GL_n(R)/SL_n(R) \cong R^*.$$