Exam 2 Solutions

1a. An isomorphism of groups is a homomorphism ($f(xy) = f(x)f(y)\ \forall x,y \in G$) which is 1-1 and onto.

b. $\text{Ker}\,\phi = \{g \in G \mid \phi(g) = e\}$

c. The root of a monic polynomial in $\mathbb{Z}[x]$.

d. If the leading coef. is 1

2a. False - $\phi(g)$ could have order 1, 2 or 4.

b. True

c. False $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$

d. False, $x^2 - 2 = (x+\sqrt{2})(x-\sqrt{2})$ in $\mathbb{R}[x]$

e. False, e.g. A_4

f. False, this thm required the polynomial be monic

g. True

h. True

i. False

j. True

3 We proved in class that $\text{Ker}\,\phi$ is a normal subgroup of G, but G is simple and $\text{Ker}\,\phi \neq G$ by assumption.

Thus $\text{Ker}\,\phi = \{e\}$. Now suppose $\phi(g_1) = \phi(g_2)$. Then

$$\phi(g_1)\phi(g_2)^{-1} = e$$
$$\phi(g_1 g_2^{-1}) \quad \text{so } g_1 g_2^{-1} \in \text{Ker}\,\phi \text{ so } g_1 g_2^{-1} = e \text{ so } g_1 = g_2$$

Thus ϕ is 1-1

4. $\phi[G]$ is abelian iff only if

$$\phi(x)\phi(y) = \phi(y)\phi(x) \quad \forall x,y \in G$$

iff only if $\phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1} = e \quad \forall x,y \in G$

" $\phi(xyx^{-1}y^{-1}) = e \quad \forall x,y \in G$ since ϕ is a homom.

iff only if $xyx^{-1}y^{-1} \in \ker\phi$ by def. of the kernel.

5a. since 1, 3, 6 have no factors in common except ± 1, we know that if $p(x)$ has a rational root, it has an integer root which divides 6. We also know, since $\deg p(x) = 3$ that $p(x)$ factors / $\mathbb{Q}$ iff only if $p(x)$ has a root in $\mathbb{Q}$.

But $p(1) = 10 \quad p(-1) = 2 \quad p(2) = 20 \quad p(-2) = -8$

$p(3) = 42, \quad p(-3) = -30 \quad p(6) = 240, \quad p(-6) = -222$

Thus p has no rational roots so $p(x)$ is irred / $\mathbb{Q}$

b. $2x^3 + 9x^2 + 3x + 6$ is irred by E.C. for $p = 3$

$p(x) = 1 \cdot q(x) + 3x-3$

$q(x) = (\frac{1}{3}x - \frac{2}{3})(3x-3) + 0$

$$\begin{array}{r|l} & \frac{1}{3}x - \frac{2}{3} \\ 3x-3 & x^2-3x+2 \\ & \underline{x^2 - x} \\ & -2x+2 \\ & \underline{-2x+2} \\ & 0 \end{array}$$

Thus a GCD = $\boxed{3x-3}$

\#7. By the Div Algorithm

$f(x) = (x-a)q(x) + r$

where r is a constant, since degree $(x-a) = 1$.

Thus

$f(a) = 0 + r = 0.$

Thus $f(a) = 0 \Rightarrow r = 0 \Rightarrow f(x) = (x-a)q(x)$ so $x-a$ is a factor.

If $x-a$ is a factor then

$f(x) = (x-a)q(x)$ so $f(a) = 0$ //

New \#7 Remainder

$\overset{\text{Remainder}}{\overset{\|}{p(-1)}} = 2+1-3-7+15 = 8$

8. $x^2 - 2$

9. $f(x) = x^3 - 2x^2 + x - 2$

$f(0) = -2$

$f(1) = -2$

$f(2) = 0$

$$\begin{array}{r} x^2 + 1 \\ x-2 \overline{)\, x^3 - 2x^2 + x - 2} \\ \underline{x^3 - 2x^2} \\ x - 2 \\ \underline{x - 2} \\ 0 \end{array}$$

$x^3 - 2x^2 + x - 2 = (x-2)(x^2+1)$

But x^2+1 has root of 2 also

$(x-2)(x+2) = x^2 - 4 \equiv x^2 + 1$

Thus

$$x^3 - 2x^2 + x - 2 = (x-2)^2(x+2)$$
$$= (x+3)^2(x+2) \text{ in } \mathbb{Z}_5[x]$$