Math 4330 Midterm Exam #2 - March 28, 2006

Instructions: Please put all work you wish graded in the blue book, you do not need to pass in the exam paper. No aids are allowed (e.g. calculators, notes, book, etc..). Cross out any work you don't wish graded and clearly label the problems if you do them out of order.

1. **(12 points)** Complete the following definitions:

a. An *isomorphism of groups* is...

b. Let $\phi : G \to H$ be a group homomorphism. The *kernel* of ϕ...

c. An *algebraic number* is....

d. A polynomial is *monic* if...

2. **(20 points)** True or false:

____ a. Suppose $\phi : G \to H$ is a group homomorphism and $g \in G$ has order 4. Then $\phi(g)$ has order 4.

____ b. Let $n \geq 2$. There exists a group homomorphism from the symmetric group S_n onto a cyclic group with two elements.

____ c. The ring of 2×2 matrices with integer entries has no zero divisors.

____ d. $p(x) = x^2 - 2$ is irreducible in $\mathbb{R}[x]$.

____ e. Any group with order 12 must contain an element of order 6.

____ f. If the polynomial $12x^3 + 16x^2 - 5x - 3$ has a rational root then it must have an integer root.

____ g. A_5 is a simple group.

____ h. The Eisenstein criterion can be applied to show that $3x^3 + 4x^2 + 4x + 2$ is irreducible over $\mathbb{Q}$.

____ i. The Eisenstein criterion can be applied to show that $3x^3 + 2x^2 + 2x + 4$ is irreducible over $\mathbb{Q}$.

____ j. If a finite group G has exactly one subgroup H of a given order then H must be normal.

3. **(12 points)** Recall that the trivial homomorphism maps every element of G to e. Let G be a simple group and $\phi : G \to H$ be a group homomorphism which is not the trivial homomorphism.. Prove that ϕ is 1-1.

4. (**12 points**) Let $\phi : G \to H$ be a group homomorphism. Prove that $\phi[G]$ is abelian if and only if for all $x, y \in G$ we have $xyx^{-1}y^{-1} \in \ker \phi$.

5. **(12 points)**
For each polynomial below $p(x)$ below, prove that $p(x)$ is irreducible over $\mathbb{Q}$. You may cite any theorems proven in class.

   a. $p(x) = x^3 + 3x + 6$

   b. $p(x) = 2x^3 + 9x^2 + 3x + 6$

6. **(10 points)** Let $p(x) = x^2 - 1$ and $q(x) = x^2 - 3x + 2$. *Use the Euclidean Algorithm* to find a gcd $d(x)$ of $p(x)$ and $q(x)$.

7. **(6 points)** What is the remainder when the polynomial $p(x) = 2x^6 - x^5 + 3x^3 - 7x^2 + 15$ is divided by $x + 1$?

8. **(6 points)** Give an example of a polynomial $f(x) \in \mathbb{Q}[x]$ which is irreducible over $\mathbb{Q}$ but not irreducible over $\mathbb{R}$.

9. **(10 points)** The polynomial $x^3 - 2x^2 + x - 2$ can be factored into linear factors in $\mathbb{Z}_5[x]$. Find this factorization.