Math 4330 Exam 1 Solutions

1. a. A ring is a set R w/ two binary operations $+, \cdot$ such that $(R,+)$ is an abelian group, $\cdot$ is associative, and $r(x+y) = rx + ry$, $(x+y)r = xr + yr$ $\forall x,y,r \in R$.

b. order of g is the smallest $n>0$ such that $g^n = \{e\}$. If no such n exists we say g has infinite order.

c. An integral domain is a commutative ring w/ identity and no zero divisors

d. σ is even if σ can be written as a product of an even # of transpositions

e. $H \trianglelefteq G$ if $gH = Hg$ $\forall g \in G$.

2. a. T b. T c. F d F e. T f. F g. F h. F

3 $\sigma = (123)(23)(145)(24)(357) = (14)(2573)(6)$

$\sigma^{-1} = (2375)(14)$

order $\sigma = 4$ σ is ~~odd~~ even

$\sigma = (17)(25)(123)(45)(17)(25) = (24)(375)$

$\sigma^{-1} = (357)(24)$

order $\sigma = 6$ σ is odd

$\sigma = (1245)(13)(1734) = (172435)$ order $\sigma = 6$

$\sigma^{-1} = (153427)$

σ is odd

In disjoint cycle notation the order is lcm of cycle lengths
In S_{15} the max is

7cycle. 3cycle. 5 cycle

order = 105

4. Let $z_1 z_2 \in Z(G)$, $g \in G$. Then $z_1 z_2 g = z_1 g z_2$ since $z_2 \in Z$
$= g z_1 z_2$ " $z_1 \in Z$

So $z_1 z_2 \in Z(G)$.

Also $g z_1 = z_1 g \Rightarrow z_1^{-1} g^{-1} = g^{-1} z_1^{-1}$ but g is arbitrary so $z_1^{-1} \in Z(G)$.

Thus $Z(G) \leq G$.

Obviously $gZ(G) = Z(G)g$ $\forall g$ since $gz = zg$ $\forall z \in Z$.
Thus $Z(G) \trianglelefteq G$.

5. a- $gg^{-1} = e \in H$ so $g \sim g$ $\forall g$

b. Suppose $g_1 \sim g_2$ so $g_1^{-1} g_2 \in H$. H is a subgroup so $(g_1^{-1} g_2)^{-1} = g_2^{-1} g_1 \in H$. Thus $g_2 \sim g_1$.

c- Suppose $g_1 \sim g_2$ and $g_2 \sim g_3$ Then $g_1^{-1} g_2 \in H$ and $g_2^{-1} g_3 \in H$.

$H \leq G$ so $g_1^{-1} g_2 \cdot g_2^{-1} g_3 = g_1^{-1} g_3 \in H$.

Thus $g_1 \sim g_3$ So $\sim$ is reflexive, symmetric & transitive.

d- The classes are left cosets gH.

6\. a. no

b. 4

c. x^3y

d. x^2

e. $Z(G) = \{e, x^2\}$

f. $\{e, xy, x^2, x^3y\}$

9\. $H = \{1, y, y^2\}$

$xH = \{x, xy, xy^2\} = Hx$

$x^2H = \{x^2, x^2y, x^2y^2\} = Hx^2$

$x^3H = \{x^3, x^3y, x^3y^2\} = Hx^3$

So $H \trianglelefteq G$.

	H	xH	x^2H	x^3H
H	H	xH	x^2H	x^3H
xH	xH	x^2H	x^3H	H
x^2H	x^2H	x^3H	H	xH
x^3H	x^3H	H	xH	x^2H

G/H is cyclic!

$H = Z(G)$ is normal by #4!!

h. $H = \{1, x^2\}$

$xH = \{x, x^3\} = Hx$

$yH = \{y, x^2y\} = Hy$

$xyH = \{xy, x^3y\} = Hxy$

$y^2H = \{y^2, x^2y^2\} = Hy^2$

$xy^2H = \{xy^2, x^3y^2\} = Hxy^2$

	H	xH	yH	xyH	y^2H	xy^2H
H	H	xH	yH	xyH	y^2H	xy^2H
xH	xH	H	xyH	yH	xy^2H	y^2H
yH	yH	xy^2H	y^2H	xH	H	xyH
xyH	xyH	y^2H	xy^2H	H	xH	yH
y^2H	y^2H	xyH	H	xy^2H	yH	xH
xy^2H	xy^2H	xyH	xH	y^2H	xyH	H

This is $\cong$ to S_3.

Think $xH = (12)$

$yH = (123)$