Math 4330 Midterm Exam #1 - February 16, 2006

1. **(15 points)** Complete the following definitions:

   a. A *ring* is...

   b. Let G be a group with $g \in G$. The *order* of g is...

   c. An *integral domain* is...

   d. A permutation $\sigma \in S_n$ is *even* if...

   e. A subgroup $H \leq G$ is *normal* if...

2. **(16 points)** True or false:

____ a. All cyclic groups are abelian.

____ b. Matrix multiplication is associative.

____ c. Suppose $H \leq G$ and H is abelian. Then H must be normal.

____ d. All groups with 6 elements are abelian.

____ e. The intersection of two subgroups of a group is always a subgroup.

____ f. The union of two subgroups of a group is always a subgroup.

____ g. The definition of a left coset of $H \leq G$ makes sense if and only if H is a normal subgroup.

____ h. The Klein 4-group is cyclic.

3. **(15 points)** Below are listed several permutations in S_7. Multiply out to write each σ in disjoint cycle notation. Then calculate the order of σ, σ^{-1} and whether σ is even or odd.

   $\sigma = (123)(23)(145)(24)(357)$

   $\sigma = (17)(25)(123)(45)(17)(25)$

   $\sigma = (1245)(13)(1734)$

   Bonus: What is the *largest* possible order of a permutation in S_{15}

4. **(15 points)**
The *center* $Z(G)$ of a group is defined to be the elements in G which commute with every element of G:

$$Z(G) = \{z \in G \mid zg = gz \forall g \in G\}.$$

For example if G is abelian then $Z(G) = G$ and if $G = S_3$ then $Z(G) = \{e\}$. Prove that $Z(G)$ is subgroup; then prove that it is normal.

5. **(15 points)** Let $H \leq G$ and define an equivalence relation $\sim$ on G by $g_1 \sim g_2$ if and only if $g_1^{-1}g_2 \in H$. Prove that $\sim$ is an equivalence relation. What are the equivalence classes?

6. **(24 points)** There are three nonisomorphic, nonabelian groups of order 12. Two of them we have already encountered, namely A_4 and D_6. Below is the Cayley table for the third.

	1	**Y**	$\mathbf{Y^2}$	**X**	**XY**	$\mathbf{XY^2}$	$\mathbf{X^2}$	$\mathbf{X^2Y}$	$\mathbf{X^2Y^2}$	$\mathbf{X^3}$	$\mathbf{X^3Y}$	$\mathbf{X^3Y^2}$
1	1	Y	Y^2	X	XY	XY^2	X^2	X^2Y	X^2Y^2	X^3	X^3Y	X^3Y^2
Y	Y	Y^2	1	XY^2	X	XY	X^2Y	X^2Y^2	X^2	X^3Y^2	X^3	X^3Y
$\mathbf{Y^2}$	Y^2	1	Y	XY	XY^2	X	X^2Y^2	X^2	X^2Y	X^3Y	X^3Y^2	X^3
X	X	XY	XY^2	X^2	X^2Y	X^2Y^2	X^3	X^3Y	X^3Y^2	1	Y	Y^2
XY	XY	XY^2	X	X^2Y^2	X^2	X^2Y	X^3Y	X^3Y^2	X^3	Y^2	1	Y
$\mathbf{XY^2}$	XY^2	X	XY	X^2Y	X^2Y^2	X^2	X^3Y^2	X^3	X^3Y	Y	Y^2	1
$\mathbf{X^2}$	X^2	X^2Y	X^2Y^2	X^3	X^3Y	X^3Y^2	1	Y	Y^2	X	XY	XY^2
$\mathbf{X^2Y}$	X^2Y	X^2Y^2	X^2	X^3Y^2	X^3	X^3Y	Y	Y^2	1	XY^2	X	XY
$\mathbf{X^2Y^2}$	X^2Y^2	X^2	X^2Y	X^3Y	X^3Y^2	X^3	Y^2	1	Y	XY	XY^2	X
$\mathbf{X^3}$	X^3	X^3Y	X^3Y^2	1	Y	Y^2	X	XY	XY^2	X^2	X^2Y	X^2Y^2
$\mathbf{X^3Y}$	X^3Y	X^3Y^2	X^3	Y^2	1	Y	XY	XY^2	X	X^2Y^2	X^2	X^2Y
$\mathbf{X^3Y^2}$	X^3Y^2	X^3	X^3Y	Y	Y^2	1	XY^2	X	XY	X^2Y	X^2Y^2	X^2

a. (2 points) Is G abelian?

b. (2 points) What is the order of the element XY?

c. (2 points) What is the inverse of the element XY?

d. (2 points) Which elements of G have order 2?

e. (2 points) What is the center Z(G)? (See #5 for definition of center.)

f. (2 points) Find a subgroup of G with 4 elements.

g. (6 points) Let $H=\{1, Y, Y^2\}$. Notice that $H \leq G$. Prove that H is a normal subgroup and give the Cayley table for the quotient group G/H. Is G/H cyclic?

h. (6 points) Let $K=\{1,X^2\}$. Notice that $K \leq G$. Prove that K is a normal subgroup and give the Cayley table for G/K. What group is G/K isomorphic to?