Proof that A_5 is simple

Theorem: The alternating group A_5 is simple.

Proof: Suppose $\{e\} \neq H \trianglelefteq A_5$. We will prove that $H = A_5$. Our first goal is to prove the following:

Lemma: H must contain a 3-cycle.

Proof: Let $\sigma \in A_5$. When we write σ in disjoint cycle notation it is either a $5-$cycle, a 3-cycle or a product of two 2-cycles. (since the other possible cycle types are odd permutations.)

Suppose H contains a 5-cycle $\sigma = (abcde)$. Since $(ace) \in A_5$ and $H \trianglelefteq A_5$ we have:

$$(ace)\sigma(ace)^{-1} = (acbed) \in H$$

But H is a subgroup so $(abcde)(acbed) = (adb) \in H$.

Now suppose H contains a product of two disjoint transpositions: $\sigma = (ab)(cd) \in H$. Then $(ae)(cd) \in A_5$ so

$$(ae)(cd)\sigma(ae)(cd)^{-1} = (be)(cd) \in H$$

since $H \trianglelefteq A_5$. Thus $(ab)(cd)(be)(cd) = (abe) \in H$.

Thus $H \neq \{e\}$ implies H contains a 3-cycle.

Now pick a 3-cycle $(abc) \in H$. Consider the following two permutations:

$$\tau = \begin{pmatrix} a & b & c & d & e \\ x & y & z & s & t \end{pmatrix}, \ \tau' = \begin{pmatrix} a & b & c & d & e \\ x & y & z & t & s \end{pmatrix}$$

Notice that $\tau(abc)\tau^{-1} = (xyz)$ and $\tau'(abc)\tau'^{-1} = (xyz)$. Exactly one of τ or τ' is in A_5 since they differ by a transposition (st). Thus $(xyz) \in H$. But (xyz) was arbitrary so H **contains all possible 3-cycles.**

Finally notice that:

$$(ab)(cd) = (dac)(abd) \text{and } (ab)(bc) = (abc).$$

Thus any even permutation can be written as a product of 3-cycles. But all possible 3-cycles are in H so H must be all of A_5, and A_5 is simple.

Remark: A_4 is not simple since $\{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup. Can you see why the proof above will not work for A_4?