Homework #14 Solutions

1 a. $f+g = 2x^2-3$

$f\cdot g = 8x^3 - 16x^2 \cancel{} +8x - 10x^2 + 20x - 10$

$= 6x^2 + 4x + 6$

b. $f+g=0$

$fg = x^2+2x+1 = x^2+1$

c. $f+g = 3x^4+2x^3+4x^2+1$

$fg = 6x^7 + 4x^4 + 8x^3 + 12x^6 + 8x^3 + 16x^2 + 9x^5 + 6x^2 + 12x + 6x^4 + 4x + 8$

$= x^7 + 2x^6 + 4x^5 + x^3 + 2x^2 + x + 3$

2. Any polynomial of degree ≤ 3 is $a_0 + a_1x + a_2x^2 + a_3x^3$ w/ 2 choices (0 or 1) for each a_i. Thus $2^4 =$ 16 polynomials

In $\mathbb{Z}_5$ there are $5^4 =$ 625 polynomials

3. a. $ev_2(3x^2+2x-5) = 11$

b. $ev_{-1}(x^3+3x) = -4$

c. $ev_5(6) = 6$

4a. $0^2+1=1$ $1^2+1=0$ roots: 1

b. $0^5+3\cdot 0^3+0^2+2\cdot 0=0$

$1^5+3\cdot 1+1+2=2$

$2^5+3\cdot 2^3+2^2+2\cdot 2 = 32+24+4+4 = 4$

$3^5+3\cdot 3^3+3^2+2\cdot 3 = 243+81+9+6 = 339 = 4$

$4^5+3\cdot 4^3+4^2+2\cdot 4 = 1024+192+16+8 = 0$

roots: 0, 4

4c. roots of $f(x)g(x)$ are roots of f and of g

roots of f: ~~1~~ 2,
roots of g: 0, 4

0, 2, 4

5. Let R be an int. domain.
$R[x]$ has an identity, namely $1 + 0x + 0x^2 + \ldots$
$R[x]$ is obviously commutative by the formula for polynomial multiplication.

Finally suppose $p(x) = a_0 + a_1 x + \ldots + a_n x^n$
$q(x) = b_0 + b_1 x + \ldots + b_m x^m$
with $a_n \neq 0$, $b_m \neq 0$. Then $a_n b_m \neq 0$ since R is an integral domain. But
$p(x)q(x) = a_n b_m x^{n+m}$ + lower degree terms

So $p(x)q(x) \neq 0$, i.e. $R[x]$ has no zero divisors

6.18

6.19 x is a factor iff only if $a_0 = 0$
Pf. by Thm 6-4, $x = x - 0$ is a factor iff ~~a_0~~ 0 is a root.
But $p(0) = a_0$ so 0 is a root iff only if $a_0 = 0$

6.23 Done in class

6.25 irr by E.C. $p=13$

6.26 If c is not div by any square, just chose a prime $p|c$ and apply E.C. for p to see x^2-c is irr.

~~Roots of x^2-c are $a \in \mathbb{Q}$ w/~~

If x^2-c factors it has a root, which is in $\mathbb{Z}$ by Thm from class. But if $a \in \mathbb{Z}$ is a root then $a^2-c=0 \Rightarrow c=a^2$.

Thus x^2-c reducible $\Rightarrow c$ is a perfect square.

6.27

x^5+2x^3+2x+2

$x^5+7x+14$

6.28 irr by EC $p=2$

1. $P(x) = x^3 + x^2 - 3x - 3 \qquad Q(x) = x^3 + 2x^2 - x - 2$

$P(x) = 1 \cdot Q(x) + (-x^2 - 2x - 1)$ $\qquad R_1 = -x^2 - 2x - 1$

$Q_1 = 1$

$$\begin{array}{r|l} & -x \\ -x^2-2x-1 & x^3+2x^2-x-2 \\ & -x^3+2x^2+x \\ \hline & -2x-2 \end{array}$$

$Q(x) = (-x)(-x^2-2x-1) + (-2x-2)$ $\qquad R_2 = -2x-2$

$= \cancel{Q}$ $\qquad Q_2 = -x$

$$\begin{array}{r|l} & \tfrac{1}{2}x + \tfrac{1}{2} \\ -2x-2 & -x^2-2x-1 \\ & -x^2-x \\ \hline & -x-1 \\ & -x-1 \\ \hline & 0 \end{array}$$

$R_1 = (\tfrac{1}{2}x + \tfrac{1}{2}) R_2 + 0$ $\qquad Q_3 = \tfrac{1}{2}x + \tfrac{1}{2} \quad R_3 = 0.$

Thus ~~the~~ a gcd is $(-2x-2)$. $(x+1)$ is another, as is any multiple.

Now to get $D(x)$ in terms of P, Q?

$\gcd = R_2 = Q - Q_2 R_1 = Q - Q_2(P - Q_1 Q)$

$= Q - Q_2 P + Q_1 Q_2 Q$

$= (1 + Q_1 Q_2) Q - Q_2 P$

$-2x-2 = (1 + -x) Q - (-x) P$

$\boxed{-2x-2 = (1-x)(x^3+2x^2-x-2) + x(x^3+x^2-3x-3)}$

2. We need 4 polynomials w/ 5 as a root

$$x-5$$
$$x^2-25$$
$$(x-5)(x-3)$$
$$(x-5)(x+1)$$

3 We need roots! Just plug in 0,1,2,...,10 into $2x^3+3x^2-7x-5$ mod 11

$$p(1) = -7 \not\equiv 0$$
$$p(2) = 9 \not\equiv 0$$
$$p(3) = 55 \equiv 0$$

Thus 3 is a root.

$$\begin{array}{r|l} & 2x^2+9x+9 \\ x-3 & 2x^3+3x^2-7x-5 \\ & 2x^3-6x^2 \\ \hline & 9x^2-7x-5 \\ & 9x^2-27x-5 \\ \hline & 9x-5 \\ & 9x-27 \\ \hline & 0 \end{array}$$

← remember en is mod 11

Check for roots of $2x^2+9x+9$

$$q(3) = 54 \not\equiv 0$$
$$q(4) = 77 \equiv 0$$

$$\begin{array}{r|l} & 2x+6 \\ x-4 & 2x^2+9x+9 \\ & 2x^2-8x \\ \hline & 6x+9 \\ & 6x-24 \\ \hline & 0 \end{array}$$

$$2x^3+3x^2-7x-5 = (x-3)(x-4)(2x+6)$$
$$= (x+8)(x+7)(2x+6) \text{ in } \mathbb{Z}_{11}$$