Homework 13 Solutions

1. Let G be finite, simple, abelian. Any subgroup of G is normal so G simple $\Rightarrow$ $\{e\}$ and G are the only subgroups.

Let $g \in G$, $g \neq e$. Then $\langle g \rangle \leq G$ so $\langle g \rangle = G$ so G is cyclic. Thus let

$$G = \{e, g, g^2, \dots, g^{k-1}\}, \quad |G| = k.$$

Suppose $k = ab$ is not prime. Then

$$\langle g^a \rangle = \{e, g^a, g^{2a}, \dots, g^{(b-1)a}\}$$

is a subgroup not $= G$.

Thus k must be prime.

2. Let $h_1 n_1$ and $h_2 n_2$ be elements of HN.

$N \trianglelefteq G$ so $\cancel{h_2}$ $\quad h_1 n_1 h_2 n_2 = h_1 (h_2 h_2^{-1} n_1 h_2 n_2)$

but $h_2^{-1} n_1 h_2 \in N$ since $N \trianglelefteq G$. Thus

$$h_1 n_1 h_2 n_2 = \underbrace{h_1 h_2}_{\in H} \cdot \underbrace{h_2^{-1} n_1 h_2 \cdot n_2}_{\in N}$$

Thus $h_1 n_1 h_2 n_2 \in HN$, HN is closed under mult.

Also $(h_1 n_1)^{-1} = n_1^{-1} h_1^{-1} = \underbrace{h_1^{-1}}_{\in H} \underbrace{h_1 n_1^{-1} h_1^{-1}}_{\in N}$

So $(h_1 n_1)^{-1} \in HN$.

Thus $HN \leq G$

3. Let $H \trianglelefteq G$, $N \trianglelefteq G$. By #N HN is a subgroup. Let $hn \in HN$.

~~But~~ Then $g(hn)g^{-1} = ghg^{-1}gng^{-1}$

and $ghg^{-1} \in H$, $gng^{-1} \in N$ since $H \trianglelefteq G$, $N \trianglelefteq G$.

Thus $g(hn)g^{-1} \in HN$, so $HN \trianglelefteq G$.

4. a. $exe^{-1} = x$ so $x \sim x$.

Suppose $x \sim y$ so $\exists\, g$ with $gxg^{-1} = y$.

Then $y = g^{-1}xg = g^{-1}x(g^{-1})^{-1}$ so $y \sim x$.

Finally suppose

$x \sim y$ and $y \sim z$.

Then $g_1xg_1^{-1} = y$ and $g_2yg_2^{-1} = z$ for some $g_1, g_2 \in G$.

Then $g_2g_1x(g_2g_1)^{-1} = g_2g_1xg_1^{-1}g_2^{-1}$

$= g_2yg_2^{-1} = z$.

Thus $x \sim z$.

So $\sim$ is an equiv. relation

b. If G is abelian then $gxg^{-1} = gg^{-1}x = x$

so each equiv. class has 1 element

c. Class 1 = $\{e\}$

Class 2 = $\{(12), (13), (23)\}$

Class 3 = $\{(123), (132)\}$

d. $H \trianglelefteq G$ if $h \in H, g \in G \Rightarrow ghg^{-1} \in H$.

Thus if $h \in H$ and $h \sim y$ then $y = ghg^{-1} \in H$.

Thus $h \in H \Rightarrow$ entire conjugacy class of h is in H.

$\therefore$ H is a union of conjugacy classes