Week 7 Solutions

4.8 Yes, reduction mod n is a homomorphism for multiplication

$$(i+rn)(j+sn) = ij + rjn + isn + rsn^2$$
$$= ij + (rj + is + rsn)\, n$$

Thus $\phi((i+rn)(j+sn)) = ij \bmod n$.

2. $\phi[G]$ is abelian if & only if
$$\phi(x)\phi(y) = \phi(y)\phi(x) \quad \forall x,y \in G.$$

But this is true if & only if
$$\underbrace{\phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1}}_{\phi(xyx^{-1}y^{-1})} = e, \quad \text{Now use homom. property}$$

Thus $\phi[G]$ is abelian if & only if $xyx^{-1}y^{-1} \in \ker \phi$.

3. Suppose ϕ is 1-1. We know $\phi(e) = e$ so $\phi(x)$ cannot be e for any other $x \neq e$. Thus $\ker\phi = \{e\}$.

Conversely let $\ker \phi = \{e\}$. To prove ϕ is 1-1 suppose $\phi(x) = \phi(y)$. Then
$$\underbrace{\phi(x)\phi(y)^{-1}}_{\phi(xy^{-1})} = e$$

Thus $xy^{-1} \in \ker\phi \Rightarrow xy^{-1} = e \Rightarrow x = y$.

Hence ϕ is 1-1. //

4. Fix $G \ni g$. Define $\phi_g : G \to G$ by $\phi_g(x) = gxg^{-1}$.

a. If $gxg^{-1} = gyg^{-1}$ then $x = y$ by cancellation. Thus ϕ is 1-1.

Let $y \in G$. Then $y = g(g^{-1}yg)g^{-1} = \phi_g(g^{-1}yg)$.

Thus ϕ_g is onto.

b. $\phi_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1}$

$= \phi_g(x)\phi_g(y)$ so ϕ_g is a homomorphism.

<u>Assigned 2/23</u>

1. Let $\phi : G \to H$ with $K = \ker \phi$. Let $k \in K$.

Notice $\phi(gk) = \phi(g)\phi(k) = \phi(g)e = \phi(g)$.

Thus $gk \in \phi^{-1}[\phi(g)]$ so $gK \subseteq \phi^{-1}[\phi(g)]$.

Now suppose $x \in \phi^{-1}[\phi(g)]$. Then

$\phi(x) = \phi(g)$ so $\phi(xg^{-1}) = e$

so $xg^{-1} \in K$ so $x \in Kg = gK$ since $K \trianglelefteq G$.

Thus $\phi^{-1}[\phi(g)] \subseteq gK$.

Hence $\phi^{-1}[\phi(g)] = gK$. //

2. $\phi: G \to G/H$ by $\phi(g) = gH$.

Let $gH \in G/H$. Then $gH = \phi(g)$ so ϕ is onto.

$g \in \operatorname{Ker}\phi$ iff $\phi(g) = e$ but the identity in G/H is H. Thus

$$\operatorname{Ker}\phi = \{g \mid gH = H\}$$

and $gH = H$ if & only if $g \in H$.

So $\operatorname{Ker}\phi = H$.