# Week 4 Solutions

3-50

| | $1$ | $a$ | $a^2$ | $a^3$ | $b$ | $ab$ | $a^2b$ | $a^3b$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $a$ | $a^2$ | $a^3$ | $b$ | $ab$ | $a^2b$ | $a^3b$ |
| $a$ | $a$ | $a^2$ | $a^3$ | $1$ | $ab$ | $a^2b$ | $a^3b$ | $b$ |
| $a^2$ | $a^2$ | $a^3$ | $1$ | $a$ | $a^2b$ | $a^3b$ | $b$ | $ab$ |
| $a^3$ | $a^3$ | $1$ | $a$ | $a^2$ | $a^3b$ | $b$ | $ab$ | $a^2b$ |
| $b$ | $b$ | $a^3b$ | $a^2b$ | $ab$ | $a^2$ | $a$ | $1$ | $a^3$ |
| $ab$ | $ab$ | $b$ | $a^3b$ | $a^2b$ | $a^3$ | $a^2$ | $a$ | $1$ |
| $a^2b$ | $a^2b$ | $ab$ | $b$ | $a^3b$ | $1$ | $a^3$ | $a^2$ | $a$ |
| $a^3b$ | $a^3b$ | $a^2b$ | $ab$ | $b$ | $a$ | $1$ | $a^3$ | $a^2$ |

2. Notice $D_4$ has 6 elements which square to identity $(e, r^2, s, sr, sr^2, sr^3)$
   Q has only 2!! They can't be $\cong$.

3-52 Look at top of page 267, $90 \circ V \neq V \circ 90$
so $V$ is not normal.

3-53 $K$ is abelian so any subgroup is normal.
Thus $V \triangleleft K$.

$[D_4 : K] = 2$ so $K$ is normal in $D_4$ (classnotes!)

But $V$ is not normal in $D_4$ by 3-52.

4. $(12) = (12)$

$(13) = (23)(12)(23)$

$(14) = (34)(23)(12)(23)(34)$

$(15) = (45)(34)(23)(12)(23)(34)(45)$

$(23) = (23)$

$(24) = (34)(23)(34)$

$(25) = (45)(34)(23)(34)(45)$

$(34) = (34)$

$(35) = (45)(34)(45)$

$(45) = 45$

5. Let $H_1 \leq G$, $H_2 \leq G$.

Let $x, y \in H_1 \cap H_2$. Then $xy \in H_1$ since $H_1 \leq G$ and $x^{-1} \in H_1$.
$xy \in H_2$ since $H_2 \leq G$, and $x^{-1} \in H_2$

Thus $xy \in H_1 \cap H_2$, and $x^{-1} \in H_1 \cap H_2$. Thus $H_1 \cap H_2 \leq G$.

6. $\{e, (12)\} \leq S_3$

$\{e, (13)\} \leq S_3$ $\quad$ $\{e, (12), (13)\}$ is Not $\leq S_3$

3-45. Let $N \triangleleft G$. Thus $gN = Ng$.

Let $n \in N$, so $gn \in gN = Ng$

Thus $gn = n'g$ for some $n' \in G$.

Thus $gng^{-1} = n' \in N$.

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a. $A_4 = \{e, (12)(34), (13)(24), (14)(23),$
$(123), (124), (134), (143), (132), (142), (234), (243)\}$

b. all but 4 elements are 3-cycles so if $H \le A_4$, $|H| = 6$
then $H$ contains at least 2 3-cycles

c. Suppose $(123) \in H$. $H$ is normal so $g(123)g^{-1} \in H$ $\forall g \in A_4$

$$(12)(34) \cdot (123) \cdot (12)(34) = (142)$$
$$(13)(24)\,(123)\,(13)(24) = (134)$$
$$(14)(23)\,(123)\,(14)(23) = (243)$$
$$\cancel{(134)\,(123)\,(142)} = \cancel{(24}$$
$$(234)\,(123)\,(243) = (134)$$
$$(243)\,(123)\,(234) = (143)$$
$$(124)\,(123)\,(142) = (243)$$
$$(142)\,(123)\,(124) = (134)$$
$$e\,(123)\,e = (123)$$

So doing $\sigma(123)\sigma^{-1}$ gives $\{(123), (134), (142), (243)\}$

But $H$ is closed under $*$ so squaring gives

$\{(132), (143), (124), (234)\}$

so $H$ contains all 8 3cycles $\neq$

1. Let $H \trianglelefteq G$, $K \trianglelefteq G$.

Let $x \in H \cap K$. Then $gxg^{-1} \in H$ $\forall g \in G$ since $H \trianglelefteq G$.

$gxg^{-1} \in K$ " " $K \trianglelefteq G$.

Thus $gxg^{-1} \in H \cap K$ $\forall g \in G$, so $H \cap K \trianglelefteq G$.

2. Let $z_1, z_2 \in Z(G)$, let $g \in G$.

$$g(z_1 z_2) = z_1 g z_2 \text{ since } z_1 \in Z(G)$$
$$= z_1 z_2 g \text{ since } z_2 \in Z(G)$$

Thus $z_1 z_2 \in Z(G)$

$g^{-1} z_1 = z_1 g^{-1}$ since $z_1 \in Z(G)$. Now take inverses:

$z_1^{-1} g = g z_1^{-1}$ $\forall g \in G$.

Thus $z_1^{-1} \in Z(G)$

So $Z(G)$ ~~⊴~~ $\leq G$.

But $gz = zg$ $\forall g \in G$, so clearly $gZ = Zg$

so $Z \trianglelefteq G$.