HW #5 Solutions

3-25 Let $S_m = \{\sigma \in S_n \mid \sigma \text{ fixes } m+1, m+2, \dots, n\}$

Clearly if $\sigma(i)=i$ and $\tau(i)=i$ then $\sigma\tau(i)=i$.
Thus $\sigma, \tau \in S_m \Rightarrow \sigma\tau$ fixes $m+1, \dots, n$. So by 3-6
$S_m \leq S_n$.

3-27 Subgroups of V

$\{1, a\}$ $\{1, a, b, c\}$
$\{1, b\}$
$\{1, c\}$
$\{1\}$

3-28 ~~Let 1_H be the i~~

Notice $eh = he = h$ $\forall h \in H$ since e is the identity for G.
But Thm 1-1 says the identity in the group is unique so
e is the identity for H.

3-29 Let $h \in G$. Since $hh^{-1} = h^{-1}h = e$ and the identity
in H is the same as in G. Thus same h^{-1} works

#2

$e \to e$ $s \to (13)$
$r \to (1234)$ $sr \to (12)(34)$
$r^2 \to (13)(24)$ $sr^2 = (24)$
$r^3 \to (1432)$ $sr^3 =$ ~~(12)~~ $(14)(23)$

is an $\cong$

3. Since $\det(AB) = \det A \det B$
and $\det(A^{-1}) = \frac{1}{\det A}$

then if A & B have determinant ± 1, clearly so does AB and A^{-1}. Thus we have a subgroup.

4. No, $e \notin C$, C is not even closed under the operation.

5.
$eV = \{e, (12)(34), (13)(24), (14)(23)\} = Ve$

$(12)V = \{(12), (34), (1324), (1423)\} = V(12)$

$(13)V = \{(13), (1234), (24), (1432)\} = V(13)$

$(14)V = \{(14), (1243), (1342), (23)\} = V(14)$

$(123)V = \{(123), (134), (243), (142)\} = V(123)$

$(124)V = \{(124), (143), (132), (234)\} = V(124)$

6.
$eW = \{e, (12), (34), (12)(34)\} = We$

~~(3~~ $(123)W = \{(123), (13), (1234), (134)\}$

$(124)W = \{(124), (14), (1243), (143)\}$

$(23)W = \{(23), (132), (234), (1342)\}$

$(24)W = \{(24), (142), (243), (1432)\}$

$(13)(24)W = \{(13)(24), (14\,23), (14)(23)\}$
(1324)

$W(123) = \{(123), (23), \cancel{(1234)}\ (1243), \cancel{(134)}\ (243)\}$

$W(13) = \{(13), (132), (143), \cancel{(143)}\ (1432)\}$

$W(14) = \{(14), (142), (134), (1342)\}$

$W(24) = \{(24), (124), (234), (1234)\}$

~~$W(243) = \{(243), (1243), (23)\}$~~

$W(1324) = \{(1324), (13)(24), (14)(23), (1423)\}$

HW # 6

3-31 Claim $g_1 \sim_R g_2$ iff $g_1 = hg_2$ for some $h \in H$ is an equiv relation.

Proof $g = e \cdot g$ and $e \in H$ so $g \sim_R g$. Thus $\sim_R$ is reflexive.

Suppose $g_1 \sim_R g_2$ so $g_1 = hg_2$. Then $g_2 = h^{-1} g_1$ and $h^{-1} \in H$ so $g_2 \sim_R g_1$. Thus $\sim_R$ is symmetric.

Finally suppose $g_1 \sim_R g_2$ and $g_2 \sim_R g_3$ so $g_1 = hg_2$ and $g_2 = h' g_3$ for some $h, h' \in H$.

$$\text{Then } g_1 = hh' g_3 \text{ so } g_1 \sim_R g_3$$

Thus $\sim_R$ is transitive.

3-39 Suppose $g^K = e$. Then $(g^{-1})^K = (g^K)^{-1} = e$. Similarly if $(g^{-1})^K = e$ then $g^K = e$.

Since the same powers of g and g^{-1} are e, the smallest such is the same, so the orders agree.

3-41 A cyclic group with n elements is of the form:

$$G = \{e, g, g^2, \ldots, g^{n-1}\} \text{ with } g^n = e$$

But $g^i g^j = g^{i+j} = g^j g^i$ so all elements of G commute.

3-44

$\sigma = (12345678)$, $(1357)(2468) = \sigma^2$, $(14725836) = \sigma^3$

$(15)(26)(37)(48) = \sigma^4$

$\sigma^5 = (16385274)$ $\sigma^6 = (1753)(2864)$

$\sigma^7 = (18765432)$ $\sigma^8 = e$

Octic group is not abelian, Cyclic groups are.

2. Order (g) = # elements in $\langle g \rangle$ and $\langle g \rangle \leq G$
so by Lagrange's Thm $|\langle g \rangle|$ divides $|G|$

3. If $[G:H]=2$ then there are 2 left cosets and two right cosets. But they are disjoint and their union is G. Since $H=eH=He$ is a left and right coset then the other left and other right coset are equal and are all elements of G not in H.

4. ~~Suppose order a = n order b = m~~
Then $a^k = e$ iff only if n divides

4. Elements of order 2 in D_4:

$$\{s, sr, sr^2, sr^3, r^2\}$$

In S_4:

(12), (13), (14), (23), (24), (34)

(12)(34), (13)(24), (14)(23)

In S_5: (12), (13), (14), (23), (24), (34), (15), (25), (35), (45)

(12)(34), (12)(35), (12)(45)

(13)(24), (13)(25), (13)(45)

(14)(23), (14)(25), ~~(14)~~ (14)(35)

(15)(23), (15)(24), (15)(34)

(23)(45), (24)(35), (25)(34)