Homework #1 Solutions

4 Suppose we add two fractions with denominators 1, 2 or 4 in lowest terms. Multiplying by $\frac{2}{2}$ or $\frac{4}{4}$ we can assume we have

$$\frac{a}{4} + \frac{b}{4} = \frac{a+b}{4}.$$ Now if $a+b$ is divisible by 2 or 4 this is not in lowest terms but the denominator in lowest terms will be 1, 2 or 4.

5. <u>Theorem</u> Let d be an integer. Then the set of fractions with denominator a divisor of d, when put in lowest terms, is closed under $+$.

<u>Proof</u> Notice that if $\frac{a}{b}$ is not in lowest terms, when we put it in lowest terms, the denominator will be a divisor of d.

Suppose x and y are divisors of d, so $d = x\,c$
$d = y\,c'$

Then $$\frac{a}{x} + \frac{b}{y} = \frac{ac}{d} + \frac{bc'}{d} = \frac{ac+bc'}{d}$$

So in lowest terms the denominator of $\frac{ac+bc'}{d}$ is a divisor of d

9. The identity is the "change" where no dancer changes position

10. The inverse of a: Position L moves to R
M moves to L
R moves to M

11. Not a group, there is no identity!

<u>#3</u>

$r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$r^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad sr = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$r^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad sr^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$r^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad sr^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

To get the multiplication table for D it saves time to notice that $srs = r^3$ and $s^2 = r^4 =$ Identity
$rs = sr^3$

<u>Mult table</u>

	e	r	r^2	r^3	s	sr	sr^2	sr^3
e	e	r	r^2	r^3	s	sr	sr^2	sr^3
r	r	r^2	r^3	e	sr^3	s	sr	sr^2
r^2	r^2	r^3	e	r	sr^2	sr^3	s	sr
r^3	r^3	e	r	r^2	sr	sr^2	sr^3	s
s	s	sr	sr^3	sr^3	e	r	r^2	r^3
sr	sr	sr^2	sr^3	s	r^3	e	r	r^2
sr^2	sr^2	sr^3	s	sr	r^2	r^3	e	r
sr^3	sr^3	s	sr	sr^2	r	r^2	r^3	e

Notice that $r^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is identity. Each element has an inverse:

$$\begin{array}{ll} e^{-1} = e & s^{-1} = s \\ r^{-1} = r^3 & (sr)^{-1} = sr \\ (r^2)^{-1} = r^2 & (sr^2)^{-1} = sr^2 \\ (r^3)^{-1} = r & (sr^3)^{-1} = sr^3 \end{array}$$

Homework #2 Solutions

1-15 θ is a binary operation, notice $a\theta b$ is just the maximum of a, b. Thus it is obviously associative, $(a\theta b)\theta c$ and $a\theta(b\theta c)$ are both the maximum of a, b, c. It is commutative.

1 is the identity.

But only 1 has an inverse.

1-17 Yes, it is commutative.

It is not the Klein 4-group, notice in the Klein 4-group that every element is its own inverse, Not true in this group.

3. We know $(g^{-1})^{-1} = g$. So write the group out in rows with each element and its inverse, e.g.

$$\begin{array}{cc} & e \\ g_1 & g_1^{-1} \\ g_2 & g_2^{-1} \\ \vdots & \\ g_k & g_k^{-1} \end{array}$$

If each $g_i \neq g_i^{-1}$ then G has $2k+1$ elements which is odd. Thus some $g_i = g_i^{-1}$!

4.

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	[illegible]	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

yes a group.

g	g^{-1}
0	0
1	7
2	6
3	5
4	4

5.

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	0	2	4	6
3	3	6	1	4	7	2	5
4	4	0	4	0	4	0	4
5	5	2	7	4	1	6	3
6	6	4	2	0	6	4	2
7	7	6	5	4	3	2	1

Not a binary operation
since not closed!

6.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

yes!

g	g^{-1}
0	0
1	6
2	5
3.	4

7.

·	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

yes, yes!

8. Thm The integers $1, 2, \dots, n-1$ form a group under multiplication modulo n if & only if n is prime.