8-3

$$\begin{tikzcd}
 & D = \mathbb{Q}(\sqrt{2}, i) \arrow[dl, dash] \arrow[dr, dash] & \\
E_1 = \mathbb{Q}(\sqrt{2}) \arrow[dr, dash] & & \mathbb{Q}(i) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Notice $D = \{a + b\sqrt{2} + ci + d\sqrt{2}\, i \mid a, b, c, d \in \mathbb{Q}\}$.

Also any $\sigma \in G(D/\mathbb{Q})$ is uniquely determined by $\sigma(\sqrt{2})$ and $\sigma(i)$.
Also $\sqrt{2}$ must map to $\pm\sqrt{2}$, i must map to $\pm i$.

So $G(D/\mathbb{Q})$ has 4 elements:

	$i \to ?$	$\sqrt{2} \to ?$
e	i	$\sqrt{2}$
σ	$-i$	$\sqrt{2}$
τ	i	$-\sqrt{2}$
$\sigma\tau$	$-i$	$-\sqrt{2}$

Notice $\sigma\tau = \tau\sigma$, $\sigma^2 = \tau^2 = e$. Thus $G(D/\mathbb{Q}) \cong V$.

Notice that E_1 is fixed by $\{e, \sigma\}$ so $G(D/E_1) \cong \{e, \sigma\}$
" " E_2 " " by $\{e, \tau\}$ so $G(D/E_2) \cong \{e, \tau\}$

It is not on the list but G has one more subgroup of two elements namely $H = \{e, \sigma\tau\}$ and it has $\mathbb{Q}(i\sqrt{2})$ as its fixed field.

8-4 We did this in 8-3

8-5 Notice $[D:E_1]=2$ since $D=\{a+bi \mid a,b\in \mathbb{Q}(\sqrt{2})\}$

$[D:E_2]=2$ " $D=\{a+b\sqrt{2} \mid a,b\in \mathbb{Q}(i)\}$

But $G(D/E_1)$ and $G(D/E_2)$ both have 2 elements, verifying Thm 8-4(ii)

8-6

$$e \subset \underset{\{e,\sigma\}}{\underset{\|}{G(D/E_1)}} \subset G(D/F)$$

$$e \subset \underset{\{e,\tau\}}{\underset{\|}{G(D/E_2)}} \subset G(D/F)$$

Inclusions are reversed because:

Suppose $F \leq E_1 \subset E_2 \leq E$.

$G(D/E_1)$ is all autos of E fixing E_1.

$G(D/E_2)$ is all autos of E fixing E_2.

If σ fixes E_2 it obviously fixes E_1 so

$$G(D/E_2) \leq G(D/E_1),$$

i.e. inclusions are reversed.

2-7 $G(D/F)$ is abelian so of course $G(D/E_1)$, $G(D/E_2)$ are normal.

$$G(D/F)/G(D/E_1) \cong \{e, \sigma, \tau, \sigma\tau\}/\{e, \sigma\} \cong \mathbb{Z}_2 \cong \cancel{G(D/E_2)}$$

$$G(E_1/F) \cong \mathbb{Z}_2 \quad \checkmark$$

2. Splitting field of $(x^2-2)(x^2-3)(x^2-5)$ is $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ is degree 8 over $\mathbb{Q}$

$$= \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{5} + e\sqrt{6} + f\sqrt{10} + g\sqrt{15} + h\sqrt{30}$$

Thus G has 8 elements. Notice

$\sqrt{2}$ can map to $\pm\sqrt{2}$
$\sqrt{3}$ " " " $\pm\sqrt{3}$
$\sqrt{5}$ can map to $\pm\sqrt{5}$

This gives 8 possibilities

	$\sqrt{2} \to ?$	$\sqrt{3} \to ?$	$\sqrt{5} \to ?$
e	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{5}$
σ	$-\sqrt{2}$	$\sqrt{3}$	$\sqrt{5}$
τ	$\sqrt{2}$	$-\sqrt{3}$	$\sqrt{5}$
ρ	$\sqrt{2}$	$\sqrt{3}$	$-\sqrt{5}$
$\sigma\tau$	$-\sqrt{2}$	$-\sqrt{3}$	$\sqrt{5}$
$\sigma\rho$	$-\sqrt{2}$	$\sqrt{3}$	$-\sqrt{5}$
$\tau\rho$	$\sqrt{2}$	$-\sqrt{3}$	$-\sqrt{5}$
[illegible]	[illegible]	$-\sqrt{3}$	$-\sqrt{5}$

Notice this is abelian and $\sigma^2 = \tau^2 = \rho^2 = e$.

Subfields

$\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{6})$, $\mathbb{Q}(\sqrt{10})$, $\mathbb{Q}(\sqrt{15})$, $\mathbb{Q}(\sqrt{30})$, $\mathbb{Q}(\sqrt{2},\sqrt{3})$,
$\mathbb{Q}(\sqrt{2},\sqrt{5})$, $\mathbb{Q}(\sqrt{3},\sqrt{5})$, $\mathbb{Q}(\sqrt{2},\sqrt{15})$, $\mathbb{Q}(\sqrt{6},\sqrt{5})$, $\mathbb{Q}(\sqrt{3},\sqrt{10})$

HW#22

Recall an arbitrary element of $\mathbb{Q}(\alpha,i)$, $\alpha=\sqrt[4]{2}$, looks like

$$X = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4 i + a_5 i\alpha + a_6 i\alpha^2 + a_7 i\alpha^3$$

$H=\{1,\sigma^3\tau\}$ where $\sigma^3\tau$ maps α to $-i\alpha$ and i to $-i$.

We set $X=\sigma^3\tau(X)$:

$$\sigma^3\tau(X) = a_0 - a_1 i\alpha - a_2\alpha^2 + a_3 i\alpha^3 - a_4 i - a_5\alpha + a_6 i\alpha^2 + a_7\alpha^3$$

So
$a_0=a_0$, $a_4=-a_4$
$a_1=-a_5$, $a_5=-a_1$
$a_2=-a_2$, $a_6=a_6$
$a_3=a_7$, $a_7=a_3$

so $a_2=0$ $a_4=0$

$$X = a_0 + a_1\alpha + a_3\alpha^3 - a_1 i\alpha + a_6 i\alpha^2 + a_3 i\alpha^3$$
$$= a_0 + a_1\alpha(1-i) + a_3\alpha^3(1+i) + a_6 i\alpha^2$$

Notice that $\dfrac{(1-i)^2}{2} = -i$

so $i = -\frac{1}{2}(1-i)^2$

Also $(i+1) = -\frac{1}{2}(1-i)^3$

So $X = a_0 + a_1\alpha(1-i) - \frac{a_3}{2}(\alpha(1-i))^3 - \frac{a_6}{2}(\alpha(1-i))^2$ we get $\mathbb{Q}(\alpha(1-i))$

HW #23

9-5

S. Field = $\mathbb{Q}(\sqrt{7})$

$G(\mathbb{Q}(\sqrt{7})/\mathbb{Q})$ is determined by where $\sqrt{7}$ maps to:

$e: a+b\sqrt{7} \to a+b\sqrt{7}$

$\sigma: a+b\sqrt{7} \to a-b\sqrt{7}$

$G(\mathbb{Q}(\sqrt{7})/\mathbb{Q}) = \{e, \sigma \mid \sigma^2 = e\}$

9-6 $x^2-4 = (x+2)(x-2)$ so

$\mathbb{Q}$ is a splitting field so

$G(\mathbb{Q}/\mathbb{Q}) = \{e\}$

9-7

$x^3-7 = (x-\sqrt[3]{7})(x-r\sqrt[3]{7})(x-r^2\sqrt[3]{7})$ where

$r^3=1$, r primitive.

Notice $E_0 = \mathbb{Q}(r)$

$E = \mathbb{Q}(r, \sqrt[3]{7})$ so elements of $G(E/E_0)$

must fix r and must take $\sqrt[3]{7}$ to $\sqrt[3]{7}$, $r\sqrt[3]{7}$

or $r^2\sqrt[3]{7}$. Let $\sigma(\sqrt[3]{7}) = r\sqrt[3]{7}$. Then

$\sigma^2(\sqrt[3]{7}) = r^2\sqrt[3]{7}$

$\sigma^3(\sqrt[3]{7}) = r^3\sqrt[3]{7} = \sqrt[3]{7}$

Thus $G(E/E_0) = \{e, \sigma, \sigma^2 \mid \sigma^3 = e\}$

9-8

$E = \mathbb{Q}(r, \sqrt[3]{7})$
$|$
$E_0 = \mathbb{Q}(r)$
$|$
$\mathbb{Q}$

We know $G(E/E_0)$ is cyclic order 3 and normal in $G(E/\mathbb{Q})$.

We know $G(E/\mathbb{Q})$ is a ~~sg~~ subgroup of S_3 since $x^3 - 7$ has 3 roots

We know $[E:\mathbb{Q}] = 6 = |G|$

Thus $G(E/\mathbb{Q})$ is a subgroup of S_3 w/ 6 elts, it is all of S_3!

10-9 $(12)(34)(123) = (243)$

$(123)(12)(34) = (134)$ and $(123), (12)(34) \in A_5$.

S_5 is not abelian so ~~A~~ $S_5 \rhd I$ is not a solv. chain