7-15 Since $i^2=-1$ and $\frac{1}{i}=-i$ and $\frac{1}{a+bi}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$

it is clear that $\mathbb{R}(i)=\{a+bi \mid a,b\in\mathbb{R}\}$

so $1, i$ form a basis

Let $q=\sqrt{2}-2i$ so $q-\sqrt{2}=2i$

$(q-\sqrt{2})^2=-4$

so $q$ is a root of $\boxed{x^2-2\sqrt{2}x+6=0}$

7-16 A splitting field of $x^4-8x^2+15$ over $\mathbb{Q}[x]$ is just $\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ where $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are the roots.

$$x^2=\frac{8\pm\sqrt{64-60}}{2}=\frac{8\pm 2}{2}=5,2$$

so $x=\pm\sqrt{5},\pm\sqrt{2}$. Thus

$\mathbb{Q}(\sqrt{2},\sqrt{5})$ is a splitting field

3a. $\alpha=\sqrt{3-\sqrt{6}}$ $\quad \alpha^2-3=-\sqrt{6}$

$(\alpha^2-3)^2=6$ $\quad \boxed{x^4-6x^2+3=0}$ degree 4

b $\alpha=\sqrt{\frac{1}{3}+\sqrt{7}}$ $\quad \alpha^2-1/3=\sqrt{7}$

$(\alpha^2-1/3)^2=7$ $\quad \boxed{x^4-\frac{2}{3}x^2-62/9=0}$ degree 4

c. $\alpha = \sqrt{2} + i$

$\alpha^2 = 2 - 1 + 2i\sqrt{2}$

$\alpha^2 - 1 = 2i\sqrt{2}$

$(\alpha^2 - 1)^2 = -8$ $\quad \boxed{x^4 - 2x^2 + 9 = 0}$ deg 4

d. $\alpha = \sqrt{2} + \sqrt{5}$

$\alpha^2 = 7 + 2\sqrt{10}$

$(\alpha^2 - 7) = 2\sqrt{10}$

$(\alpha^2 - 7)^2 = 40$ $\quad \boxed{x^4 - 14x^2 + 9 = 0}$ deg 4

4.

a. algebraic $p(x) = x^2 + 1$ degree $= 2$

b. transcendental

c. algebraic $p(x) = x - \pi^2$ degree $= 1$

d. algebraic $p(x) = x^3 - \pi^6$ degree $= 3$

5. a. $\mathbb{Q}(\sqrt{2}) : \mathbb{Q}$ degree 2 basis $\{1, \sqrt{2}\}$

b. $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{18}) : \mathbb{Q}$ degree 4 basis $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$

c. $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) : \mathbb{Q}$ degree 6 basis $\{1, \sqrt[3]{2}, \sqrt[3]{4}, \sqrt{3}, \sqrt{3}\sqrt[3]{2}, \sqrt{3}\sqrt[3]{4}$

d. $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$ (class notes)

so degree $= 1$

e. $\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}(\sqrt{3})$ degree 2 basis $\{1, \sqrt{2}\}$

6. $\alpha = \sqrt{2}+\sqrt{3}$

minimal polynomial is $x^4 - 10x^2 + 1 = 0$

roots are $x^2 = \dfrac{10 \pm \sqrt{100-4}}{2}$

$x^2 = \dfrac{10 \pm 4\sqrt{6}}{2}$

$= 5 \pm 2\sqrt{6}$

$\underbrace{x = \pm\sqrt{5 \pm 2\sqrt{6}}}_{4 \text{ conjugates}}$

note that $\alpha = \sqrt{2}+\sqrt{3} = \sqrt{5+2\sqrt{6}}$

$\alpha = \sqrt[3]{2}$, min poly is $x^3 - 2$

conjugates $\sqrt[3]{2}, \sqrt[3]{2}e^{2\pi i/3}, \sqrt[3]{2}e^{4\pi i/3}$

# HW#20

1. This is part of Thm 8-1, proof on p 143

2. Let $\sigma \in G(\mathbb{C}/\mathbb{R})$. Since $\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\}$ and $\sigma$ fixes $\mathbb{R}$, it is clear that $\sigma$ is completely determined by $\sigma(i)$.

$$\text{But} \quad \underset{\sigma(i)\sigma(i)}{\sigma(i^2)} = \sigma(-1) = -1$$

so $\sigma(i)^2$ must be $-1$. Thus $\sigma(i) = \pm i$. Hence

$$G(\mathbb{C}/\mathbb{R}) = \{\text{identity},\ a+bi \to a-bi\} \quad \text{two elements}$$

3. char $\mathbb{Q} = 0$ char $\mathbb{Z}_7 = 7$

Thm char must be 0 or prime.

Proof Suppose char $F = n = a \cdot b$ w/ $a,b > 1$. Then

$$0 = \underbrace{1+1+\cdots+1}_{n\text{ times}} = \underbrace{(1+1+\cdots+1)}_{a\text{ times}}\underbrace{(1+1+\cdots+1)}_{b\text{ times}}$$

by distr. law. Hence $\underbrace{(1+1+\cdots+1)}_{a\text{ times}}$ or $\underbrace{(1+1+\cdots+1)}_{b\text{ times}}$ must be 0, since $F$ has no zero divisors.

This contradicts $n$ being the smallest.

Thus $n$ is prime.