HW #6 Solutions

1. $x+2$ is a factor iff and only if -2 is a root.

$$f(-2) = 16 - 8 + 4 + 2 + 1 = 15$$

$$15 \equiv 0 \bmod p \text{ iff } p = 3 \text{ or } 5$$

2. constant term can't be zero else 0 is a root.

poly	root
x^3	0
x^3+1	2
x^3+2	1
x^3+x+1	1
x^3+x+2	2
x^3+2x+1	None
x^3+2x+2	None
x^3+x^2+1	~~None~~ 1
x^3+x^2+2	None
x^3+2x^2+1	None
x^3+2x^2+2	2
x^3+x^2+x+1	2
x^3+x^2+x+2	None
x^3+x^2+2x+1	None
x^3+x^2+2x+2	1
x^3+2x^2+x+1	None
x^3+2x^2+x+2	1
x^3+2x^2+2x+1	1
x^3+2x^2+2x+2	None

So x^3+2x+1, x^3+2x+2, x^3+x^2+2, x^3+2x^2+1, x^3+x^2+x+2, x^3+x^2+2x+1, x^3+2x^2+x+1, x^3+2x^2+2x+2

and $2\cdot$ each of those

16 irreducible cubics

3 x^4-22x^2+1 has no roots in $\mathbb{Q}$ since ± 1 are not roots. Thus if it factors it must be into 2 quadratics. If it factors over $\mathbb{Q}$ it factors over $\mathbb{Z}$ by Gauss Lemma.

$(x^2+ax+b)(x^2+cx+d) = x^4-22x^2+1$ gives

$$d+b=0$$
$$d+ac+b=-22$$
$$ad+bc=0$$
$$bd=1$$

Thus $b=d=\pm 1$ but $d+b=0 \Rightarrow d=b=0$.

This is a $\divideontimes$ so x^4-22x^2+1 is irred / $\mathbb{Q}$.

4 $G/\ker\phi \cong H$ by Fund. Hom. Thm

Thus $|H| = \frac{|G|}{|\ker\phi|}$ so $|\ker\phi| = 15$. Now ϕ is onto so $h=\phi(g)$ for some g.

~~We proved~~

Thus $\phi^{-1}(h) = \phi^{-1}(\phi(g)) = gK$ by HW problem.

Thus # elts mapping to h $= |\phi^{-1}(\phi(g))| = |gK| = |K| = 15$

5. $\alpha = \sqrt{2} + \sqrt{3}$

$$\alpha^2 = 5 + 2\sqrt{6}$$

$$\alpha^2 - 5 = 2\sqrt{6}$$

$$(\alpha^2 - 5)^2 = 24$$

So α is a root of $(x^2-5)^2 - 24 = 0$.

$\alpha = \sqrt{\sqrt[3]{2} - i}$

$$\alpha^2 = 2^{1/3} - i$$

$$\alpha^2 - 2^{1/3} = -i$$

$$(\alpha^2 - 2^{1/3})^2 = -1$$

$$\alpha^4 - 2^{4/3}\alpha^2 + 2^{2/3} = -1$$

$$\alpha^4 = 2^{4/3}\alpha^2 + 2^{2/3} - 1$$

$$2^{1/3}\alpha^4 = 2^{5/3}\alpha^2 + 2 - 2^{1/3}$$

$$\alpha^2 + i = 2^{1/3}$$

$$(\alpha^2 + i)^3 = 2$$

$$\alpha^6 + 3i\alpha^4 - 3\alpha^2 - i = 2$$

$$\alpha^6 - 3\alpha^2 - 2 = 3i\alpha^4 + i$$

$$\cancel{(\alpha^6 - 3\alpha^2 - 2)^2 = -9\alpha^8 - 6\alpha^4}$$

$$\alpha^6 - 3\alpha^2 - 2 = i(3\alpha^4 + 1)$$

$$(\alpha^6 - 3\alpha^2 - 2)^2 = -(3\alpha^4 + 1)^2$$

so α is a root of

$$(x^6 - 3x^2 - 2)^2 + (3x^4 + 1)^2 = 0$$