Homework # 16- Due Tuesday 3/28/06, Assigned 3/21/06

True or False (Not to be handed in)

____ a. The intersection of two normal subgroups of G is also normal..

True

____ b. There are no abelian simple groups of order 15.

True: The only abelian simple groups are cyclic of prime order, as we proved on the homework.

____ c. $\mathbb{C}$ is an extension field of $\mathbb{R}$.

True

____ d. $3x^3 + 6x^3 + 12x + 24$ is irreducible over $\mathbb{Q}[x]$ by the Eisenstein criterion.

False: There is no prime that works, 3 does not work since it divides the leading coefficient, 2 does not work since $2^2 = 4$ divides the constant term.

____ e. The division algorithm works in the ring $\mathbb{Z}_{12}[x]$.

False: We have only used the division algorithm in $F[x]$ where F is a field. There are slightly more general rings R for which the division algorithm works in $R[x]$ but at a minimum we need R to be an integral domain, and $\mathbb{Z}_{12}$ is not.

____ f. If $f(x) \in F[x]$ has a root in F and degree ≥ 2 then it cannot be irreducible over F.

True: If $a \in F$ is a root then $x - a$ is a factor by the Factor Theorem, so $f(x)$ is not irreducible.

____ g. If $f(x)$ is reducible in $F[x]$ then it must have a root in F.

False: This would only be true if we assumed $f(x)$ had degree 2 or 3. For example $f(x) = (x^4 + 2x^2 + 1) = (x^2 + 1)(x^2 + 1)$ but $f(x)$ has no roots in $\mathbb{R}$.

____ h. Let $f : G \to H$ be a group homomorphism. If $g \in G$ and $g^4 = e$ then $f(g)^4 = e$.

True: Since f is a homomorphism then $f(g^n) = f(g)^n$ and $f(e) = e$. Combining these two says $e = f(e) = f(g^4) = f(g)^4$.

____ i. Let $f : G \to H$ be a group homomorphism. If $g \in G$ has order 4 then $f(g)$ has order 4.

False: Compare this to h. above. Just because $f(g)^4 = e$ does not mean $f(g)$ has order 4, the order could be 1, 2 or 4. For instance maybe $f(g) = e$

____ j. If $3x^3 + 2x^2 - 5x + 15$ has a rational root then it must be have an integer root which divides 15.

False: The relevant theorem here required the leading coefficient to be 1.

____ k. Let G be a group with 60 elements. Then G must have an element of order 4.

False: Cauchy's theorem only guarantees an element of order p when p is a *prime* dividing the order of the group.

____ l. Let G be a nonabelian simple group. Then the center of G contains only one element.

True: The center $Z(G)$ is always normal. Since G is nonabelian, $Z(G) \neq G$ so it must be that $Z(G) = \{e\}$.

____ m. Let $n = ab$ with $a, b \in \mathbb{Z}$. If $6 \mid ab$ then $6 \mid a$ or $6 \mid b$.

False: The statement would hold if 6 were replaced by a prime number.

____ n. $x^2 + 1$ is irreducible over $\mathbb{C}$.

False: $x^2 + 1 = (x - i)(x + i)$. It is true that $x^2 + 1$ is irreducible *over* $\mathbb{R}$.

____ o. A_4 is a simple group.

False: The subgroup $\{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of A_4. It is true that A_n is simple for $n \geq 5$.

____ p. It is possible to have a homomorphism from a group with 60 elements to a group with 25 elements which is onto.

False: Let $f : G \to H$ be the homomorphism. By the fundamental theorem on homomorphisms we have:

$$G/\ker f \cong \text{Image}\, f$$

The number of elements in $G/\ker f$ is $|G|/\ker f$ which must be a divisor of 60. The image is H which has 25 elements, giving a contradiction.

____ q. If R has zero divisors then so does $R[x]$.

True: If $ab = 0$ in R then the constant polynomials a and b are in $R[x]$ and ab is still zero.

____ r. Any cubic polynomial in $\mathbb{Q}[x]$ must have a root in $\mathbb{Q}$.

False: Calculus can be used to show it must have a *real* root.