Name:

Math 3820- Midterm Exam #3 - April 18, 2005

1. **(15 points)** For each $F(s)$ calculate the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

a.

$$F(s) = \frac{4}{(\frac{s}{12} - 3)^2 + 9}.$$

b.

$$F(s) = 5 + \frac{d^5}{ds^5}(\frac{s}{s^2 - 4})$$

c.

$$F(s) = \frac{s - 4}{(s - 3)^2 + 4}$$

2. (**10 points)** Find the Laplace transform of:

$$f(t) = \int_0^t e^{(2(t-v)} sinh(v) dv$$

3. **(25 points)** Find the solution of the initial value problem (your solution may not include any convolution integrals).

$$y'' + 2y' + 2y = 1 - u_\pi(t); \;\; y(0) = 1, \;\; y'(0) = -1.$$

4. (**15 points**) Determine $\mathcal{L}\{y\}$ where y is a solution to the initial value problem. You do not need to solve for y.

$$y'' + 4y = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < \infty; \end{cases} \quad y(0) = 0, \;\; y'(0) = 0.$$

5. **(15 points)** Match the seven initial value problems below with the graphs of their solution on the attached sheet.

____ a. $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi),\ y(0) = 0,\ y'(0) = 0$

____ b. $y'' + 3y' + 4y = \delta(t - \pi) - \delta(t - 2\pi),\ y(0) = 0,\ y'(0) = 0$

____ c. $y'' + y' + 4y = \delta(t - \pi) - \delta(t - 2\pi),\ y(0) = 0,\ y'(0) = 0$

____ d. $y'' + y' + 4y = \delta(t - \pi) - \delta(t - 2\pi),\ y(0) = 0,\ y'(0) = 2$

____ e. $y'' + 4y' + 4y = \delta(t - \pi) - \delta(t - 2\pi) + \cos(t),\ y(0) = 0,\ y'(0) = 0$

____ f. $y' + y = f(t),\ \ y(0) = 0,\ \ y'(0) = 0$ where:

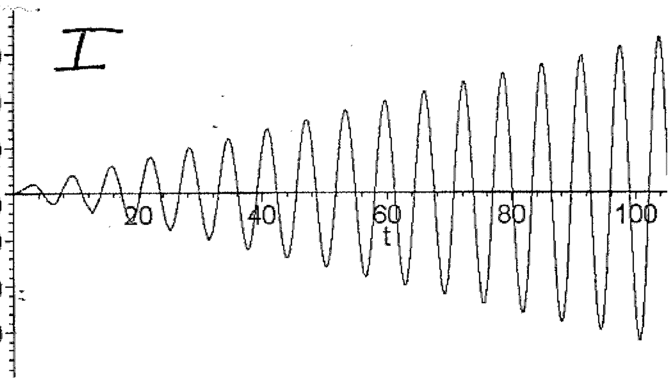
$$f(t) = u_0(t) + 2\sum_{k=1}^{100}(-1)^k u_{k\pi}(t).$$

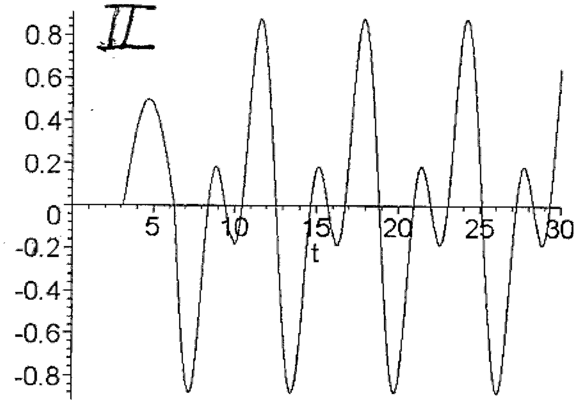
____ g. $y' + y = f(t),\ \ y(0) = 0,\ \ y'(0) = 0$ where:

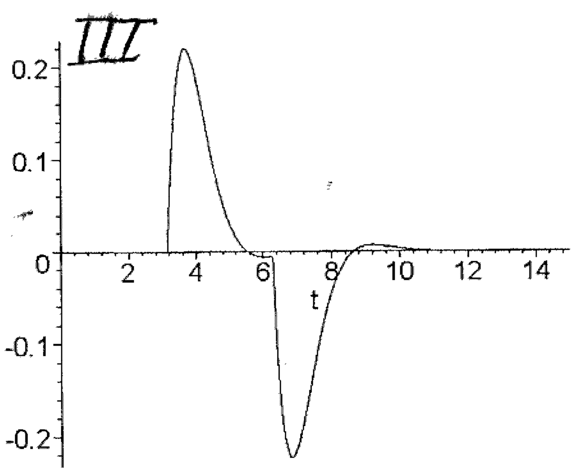
$$f(t) = u_0(t) + 2\sum_{k=1}^{100}(-1)^k u_{11*k/4}(t).$$

6. **(15 points)** Give the definition of the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$. Give an example of a function whose Laplace transform does not exist.

7. **(5 points)** Explain why $\cos(s)$ cannot be the Laplace transform of a function. You may cite any theorem proved in class or on the homework.


I
30
20
10
0
-10
-20
-30
20
40
60
80
100
t


II
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
5
10
15
20
25
30
t


III
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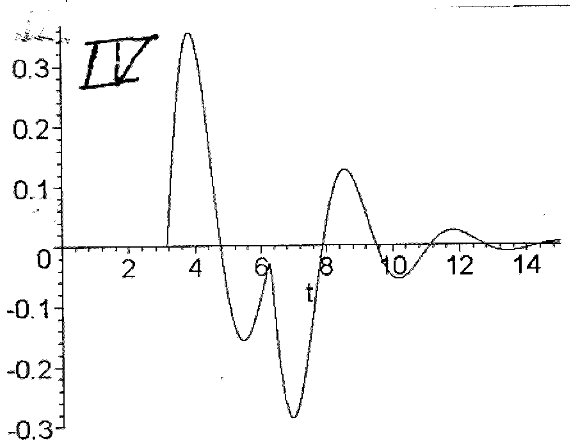
0.1
0
-0.1
-0.2
2
4
6
8
10
12
14
t


IV
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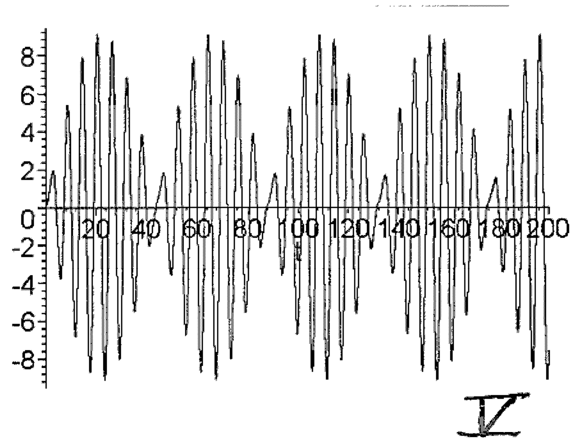
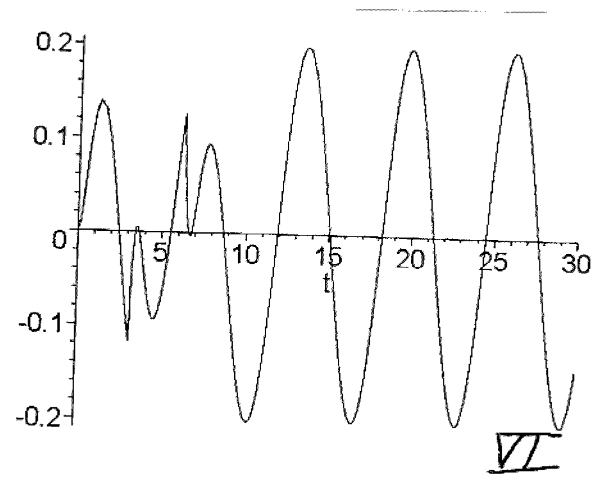
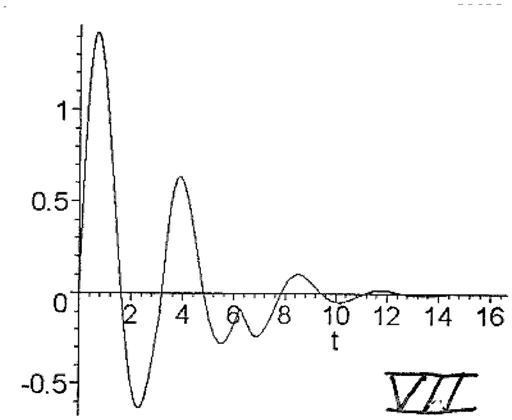
0.2
0.1
0
-0.1
-0.2
-0.3
2
4
6
8
10
12
14
t


8
6
4
2
0
-2
-4
-6
-8
20
40
60
80
100
120
140
160
180
200
t
V


0.2
0.1
0
-0.1
-0.2
5
10
15
20
25
30
t
VI


1
0.5
0
-0.5
2
4
6
8
10
12
14
16
t
VII

1.

a. $\mathcal{L}^{-1}\left\{\frac{4}{(u-3)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{3}\cdot\frac{3}{(u-3)^2+3^2}\right\} = \frac{4}{3}e^{3t}\sin(3t)$

so line 15 gives $F(\frac{s}{c}) \to cf(ct)$, here $c=12$

$\frac{4}{3}\cdot 12 = 16$ $\boxed{16e^{36t}\sin(36t)}$

b. $\mathcal{L}^{-1}\left(\frac{s}{s^2-4}\right) = \cosh 2t$ so using line 19 & 17

$\boxed{5\delta(t) - t^5\cosh(2t)}$

c. $F(s) = \frac{s-3}{(s-3)^2+4} - \frac{1}{(s-3)^2+4}$

$\boxed{e^{3t}\cos 2t - \frac{1}{2}e^{3t}\sin 2t}$

2. $f(t) = e^{2t} * \sinh t$

$\mathcal{L}\{f\} = \boxed{\frac{1}{s-2}\cdot\frac{1}{s^2-1}}$

$$s^2\mathcal{L}(y) - s + 1 + 2(s\mathcal{L}(y) - 1) + 2\mathcal{L}(y) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$(s^2+2s+2)\mathcal{L}(y) - s - 1 = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$\mathcal{L}(y) = \frac{1+s}{s^2+2s+2} + \frac{1}{s(s^2+2s+2)} - \frac{1}{s(s^2+2s+2)}e^{-\pi s}$$

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2} \qquad 1 = A(s^2+2s+2) + (Bs+C)s$$

$$1 = 2A \qquad A = 1/2$$

$$0 = 2A + C \qquad C = -1$$

$$0 = A + B \qquad B = -1/2$$

$$\mathcal{L}\{y\} = \frac{1+s}{s^2+2s+2} + \left(\frac{1/2}{s} + \frac{-\frac{1}{2}s - 1}{s^2+2s+2}\right)(1 - e^{-\pi s})$$

$$= \frac{1+s}{(s+1)^2+1} + \left(\frac{1}{2}\cdot\frac{1}{s} + \frac{-\frac{1}{2}s - \frac{1}{2} - \frac{1}{2}}{s^2+2s+2}\right)(1 - e^{-\pi s})$$

$$= \frac{1+s}{(s+1)^2+1} + \left(\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s+1}{(s+1)^2+1} - \frac{1/2}{(s+1)^2+1}\right)(1 - e^{-\pi s})$$

$$y = e^{-t}\cos t + \left(\frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t\right) - u_\pi(t)\left(\frac{1}{2} - \frac{1}{2}e^{-(t-\pi)}\cos(t-\pi) - \frac{1}{2}e^{-(t-\pi)}\sin(t-\pi)\right)$$

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4. $y'' + 4y = t + u_1(t)(1-t)$

$= t - u_1(t)(t-1)$

$(s^2+4)\mathcal{L}(y) = \frac{1}{s^2} - e^{-s}\frac{1}{s^2}$

$\mathcal{L}(y) = \frac{1}{s^2(s^2+4)}(1-e^{-s})$

5. a. $\underline{II}$ b. $\underline{III}$ c. $\underline{IV}$ d. $\underline{VII}$

e. $\underline{VI}$ f. $\underline{I}$ g. V

6. $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$\mathcal{L}\{\tan t\}$ does not exist.

7. We proved on HW that $\lim_{s\to\infty} F(s) = 0$.

This is not true for $\cos s$.