Name: SOLUTIONS

Math 3820- Midterm Exam #2 - March 18, 2005

1. **(25 points)**

a. Find a general solution for:

$$y'' + 2y' + 10y = \sin(3t).$$

b. Describe with a rough sketch and/or with words the behavior of the solutions as $t \to \infty$.

a. $r^2+2r+10=0 \qquad r=\frac{-2\pm\sqrt{-36}}{2} = \frac{-2\pm 6i}{2} = -1 \pm 3i$

Homog solutions:

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)$$

Guess: $y(t) = A\sin(3t) + B\cos(3t)$

$y'(t) = 3A\cos(3t) - 3B\sin(3t)$

$y''(t) = -9A\sin(3t) - 9B\cos(3t)$

$$y''+2y'+10y = (-9A-6B+10A)\sin(3t) + (-9B+6A+10B)\cos(3t)$$

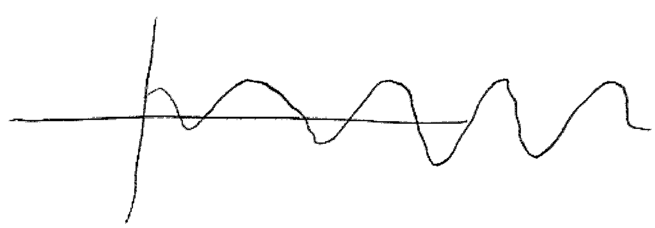
So $A-6B=1$

$B+6A=0 \qquad B=-6A \qquad 37A=1$

$A = 1/37 \qquad B = -6/37$

$$\boxed{y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{1}{37}\sin(3t) - \tfrac{6}{37}\cos(3t)}$$

b. As $t\to\infty$ it converges to oscillation $\frac{1}{37}\sin(3t) - \frac{6}{37}\cos(3t)$



2. **(15 points)** Assume that $p(t)$ and $q(t)$ are continuous. Also assume that $y_1(t)$ and $y_2(t)$ are solutions to the differential equation $y'' + p(t)y' + q(t)y = 0$. Suppose there is a point t_0 with $y_1(t_0) = 0$ and $y_2(t_0) = 0$. Explain why y_1 and y_2 cannot be a fundamental set of solutions.

If Y_1 & Y_2 are fundamental then any solution is of the form $c_1 Y_1 + c_2 Y_2 = Y$

But then $y(t_0) = c_1 Y_1(t_0) + c_2 Y_2(t_0) = 0+0 = 0$

However the E. & U. thm guarantees a solution for <u>any</u> initial condition $y(t_0)$ so if $y(t_0) \neq 0$ we can't get a solution $c_1 Y_1 + c_2 Y_2$.

3. **(15 points)** Find a general solution:

$$y'' - 6y' + 9y = 0$$

$r^2 - 6r + 9 = 0 \quad (r-3)^2 = 0$

$$y = c_1 e^{3t} + c_2 t e^{3t}$$

4. **(25 points)** Consider

$$t^2y'' + 3ty' + y = 0.$$

a. Use Abel's theorem to find the Wronskian $W(y_1, y_2)$ for a pair of fundamental solutions y_1 and y_2.

b. Verify that $y_1(t) = t^{-1}$ is a solution.

c. Use parts a. and b. to find a second fundamental solution by solving the appropriate first order differential equation.

a. $y'' + \frac{3}{t}y' + \frac{1}{t^2}y = 0$

$W(y_1, y_2) = Ce^{-\int \frac{3}{t}dt} = Ce^{-3\ln|t|} = Ce^{\ln(\frac{1}{t^3})}$

$= \frac{C}{t^3}$

So $\begin{vmatrix} t^{-1} & y_2 \\ -t^{-2} & y_2' \end{vmatrix} = Ct^{-3}$

b $y_1 = t^{-1}$ $y_1' = -t^{-2}$ $y_1'' = 2t^{-3}$

$t^2y'' + 3ty' + y = t^2 2t^{-3} + 3t \cdot (-t^{-2}) + t^{-1}$

$= 2t^{-1} - 3t^{-1} + t^{-1} = 0.$

c. $y_2' t^{-1} + y_2 t^{-2} = Ct^{-3}$) times t^2 on both sides

$ty_2' + y_2 = Ct^{-1}$

$(ty_2)' = Ct^{-1}$

$ty_2 = C_1\ln|t| + C_2$

$y_2 = C_1\frac{\ln t}{t} + C_2 t^{-1}$

So $\boxed{y_2(t) = \frac{\ln t}{t}}$

5. **(20 points)** A mass weighing 2 kg stretches a spring 10 cm. Suppose the mass is pushed upward, contracting the spring 4cm, and then set in motion with a downward velocity of 6 cm/sec. If there is no damping, determine the position of the mass at anytime t. (Remember the acceleration due to gravity is 9.8 m/sec^2).

$mg = kL$ $\qquad 2\,kg \cdot \frac{9.8\,m}{sec^2} = k \cdot .1\,m$ $\qquad k = 196 \frac{kg}{sec^2}$

$mu'' + \gamma u' + ku = 0$

$2u''(t) + 196u(t) = 0$

$u(0) = -.04 \qquad u'(0) = .06$

$u'' + 98u = 0$

$r^2 + 98 = 0 \qquad r = \pm\sqrt{98}\,i = \pm 7\sqrt{2}\,i$

$u = c_1 \cos(7\sqrt{2}\,t) + c_2 \sin(7\sqrt{2}\,t)$

$u' = -7\sqrt{2}\,c_1 \sin(7\sqrt{2}\,t) + 7\sqrt{2}\,c_2 \cos(7\sqrt{2}\,t)$

$-.04 = c_1$

$.06 = 7\sqrt{2}\,c_2 \qquad c_2 = \frac{.06}{7\sqrt{2}}$

$u(t) = -.04\cos(7\sqrt{2}\,t) + \frac{.06}{7\sqrt{2}} \sin(7\sqrt{2}\,t)$