Name: SOLUTIONS

Math 3820- Midterm Exam #1 - February 11, 2005

1. **(20 points)** Solve the initial value problem below:

$$2y + 2\sin y + (x + x\cos y)\frac{dy}{dx} = 0, \quad y(1) = \pi/2$$

$M_y = 2 + 2\cos y \quad N_x = 1 + \cos y$ so it's not exact, we use an int. factor

$\mu(2y + 2\sin y) + \mu(x + x\cos y)\frac{dy}{dx} = 0$

Need: $\mu_y(2y + 2\sin y) + \mu(2 + 2\cos y) = \mu_x(x + x\cos y) + \mu(1 + \cos y)$

Suppose $\mu(x,y) = \mu(x)$ so $\mu_y = 0$. Then

$\mu(1 + \cos y) = \mu_x \cdot x(1 + \cos y)$

$\mu = x\mu_x$

$\frac{1}{x} = \frac{1}{\mu} \cdot \mu_x$

$\ln|x| = \ln|\mu| \quad \boxed{\mu = x}$

$2xy + 2x\sin y + (x^2 + x^2\cos y)\frac{dy}{dx} = 0$ Now it's exact.

$\psi(x,y) = \int 2xy + 2x\sin y \, dx = x^2y + x^2\sin y + h(y)$

$\psi(x,y) = \int x^2 + x^2\cos y \, dy = x^2y + x^2\sin y + h(x)$

$x^2y + x^2\sin y = C$

$1^2 \cdot \pi/2 + 1^2 \cdot 1 = C \qquad C = 1 + \pi/2$

$$\boxed{x^2y + x^2\sin y = 1 + \pi/2}$$

2. **(20 points)** Find the general solution of the differential equation and describe the behavior of the solutions as $t \to \infty$.

$$ty' + 2y = \cos t \quad t > 0.$$

$y' + \frac{2}{t} y = \frac{\cos t}{t}$ use I.F.

Need $\mu' = \frac{2}{t}\mu$

$\frac{\mu'}{\mu} = \frac{2}{t}$ $\ln|\mu| = 2\ln|t| = \ln t^2$

$\mu = t^2$

$(t^2 y)' = t\cos t$

$t^2 y = \cancel{t\sin t + \cos t} + C$

$y = \frac{\sin t}{t} + \frac{\cos t}{t^2} + \frac{C}{t^2}$

As $t \to \infty$ This solution $\to 0$.

3. (**20 points**) Suppose car loans have an interest rate of 7.2 % compounded continuously and suppose I can afford a payment of \$400 per month. If the loan is to be for 5 years and I am borrowing the entire amount, how expensive a car can I afford?

Let $P(t)$ = amount I owe, t in months

Given $P(60) = 0$ what is $P(0)$?

monthly rate $= \frac{.072}{12} = .006$

$$\frac{dP}{dt} = .006P - 400$$

$$P' - .006P = -400$$

$$(e^{-.006t}P)' = -400e^{-.006t}$$

$$e^{-.006t}P = 66666.7e^{-.006t} + C$$

$$P = 66666.7 + Ce^{+.006t}$$

Given $0 = 66666.7 + Ce^{+.006 \cdot 60}$

$$C = \frac{-66666.7}{e^{+.36}} = \cancel{-93555.34}\ 46511.8$$

$P(t) = \cancel{66666.7 - 93555}$

$$66666.7 - 46511.8e^{.006t}$$

$P(0) = 66666.7 - 46511.8 =$

\$20,155

4. **(20 points)** Explain the difference between linear and nonlinear equations as relates to the existence and uniqueness of solutions to the initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0$$

If we have a linear equation $y' + p(t)y = g(t)$ then as long as $p(t)$ & $g(t)$ are continuous in an interval around t_0 There is a unique solution and it exists for as long as $p(t)$ & $g(t)$ are continuous

If it is nonlinear we need f and $\frac{\partial f}{\partial y}$ to be continuous on a neighborhood of (t_0, y_0). In this case there is a unique solution but it is only guaranteed to exist for some $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

5. **(20 points)**

$$\frac{dy}{dt} = y(y-1)(y-3).$$

Sketch the direction field, find all equilibrium solutions and classify them as stable or unstable. Also sketch in the solution corresponding to the initial condition $y(0) = 2$.

eq. solutions $y=0$ $y=1$ $y=3$

stable

unstable