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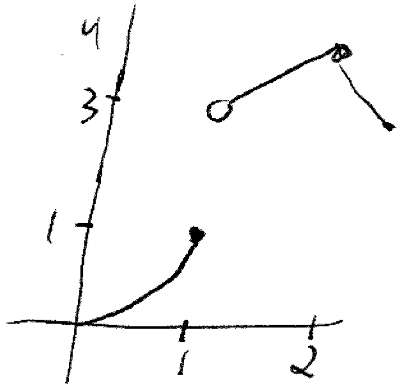
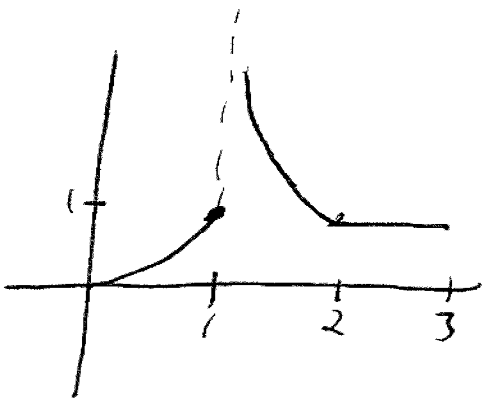
1.



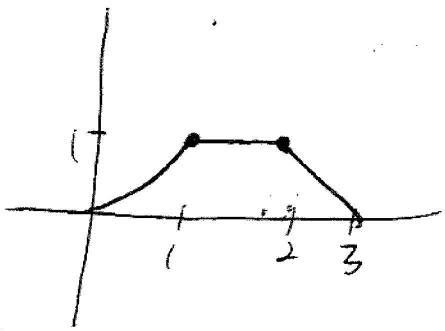
piecewise

2.



neither

3



continuous

5.

a. $\mathcal{L}\{t\} = \frac{1}{s^2} \quad s>0$ (Done in Class)

6. $\int_0^\infty e^{-st}\cos(at)\,dt = \lim_{A\to\infty}\left(\frac{1}{a^2+s^2}\left(-se^{-st}\cos(at)+ae^{-st}\sin at\right)\right)\Big|_0^A$ &

using maple

$$= \lim_{A\to\infty} \frac{1}{a^2+s^2}\left(-se^{-As}\cos(aA)+ae^{-As}\sin aA + s\cos 0 - a\sin 0\right)$$

$$= \boxed{\frac{s}{s^2+a^2}}$$

7. $\int_0^\infty e^{-st}\cosh bt\,dt = \int_0^\infty e^{-st}\left(e^{bt}+e^{-bt}\right)/2\,dt$

$$= \int_0^\infty \tfrac{1}{2}e^{(b-s)t}+\tfrac{1}{2}e^{(-b-s)t}\,dt$$

$$= \tfrac{1}{2}\left(\frac{1}{s-b}+\frac{1}{s+b}\right) = \frac{1}{2}\cdot\frac{2s}{s^2-b^2}$$

$$= \boxed{\frac{s}{s^2-b^2}}$$

21. $\int_0^\infty \frac{1}{t^2+1}\,dt$ converges (compare to $\frac{1}{t^2}$)

22. $\int_0^\infty \frac{t}{e^t}\,dt$ converges

23. $\int_1^\infty t^{-2}e^{t}\,dt$ diverges

24. $\int_0^\infty e^{-t}\cos t\,dt$ converges

25. $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$u = f(t) \qquad v = -\frac{1}{s}e^{-st}$

$du = f'(t)dt \quad dv = e^{-st}dt$

$$\mathcal{L}\{f(t)\} = -\frac{1}{s}e^{-st}f(t)\Big|_0^\infty + \frac{1}{s}\int e^{-st} f'(t)\,dt$$

$$= 0 - -\frac{1}{s} + \frac{1}{s}\int_0^\infty e^{-st} f'(t)\,dt$$

$$= \frac{1}{s} + \frac{1}{s}\int_0^\infty e^{-st} f'(t)\,dt$$

Now once s is $> a$ where a comes from $f'(t)$ being exponentially bounded then:

$$\int_0^\infty e^{-st} f'(t)\,dt \le \int_0^\infty e^{-st} e^{at}\,dt = \frac{1}{\cancel{a-s}\, s-a}$$

Thus $\mathcal{L}\{f(t)\} \le \frac{1}{s} + \frac{1}{s}\left(\frac{1}{\cancel{a-s}\, s-a}\right)$ which $\Rightarrow 0$ as $s \to \infty$.

26. $\Gamma(p+1) = \int_0^\infty e^{-x} x^p\,dx$

Let $u = x^p \qquad v = -e^{-x}$

$du = px^{p-1}dx \quad dv = e^{-x}dx$

$$\Gamma(p+1) = -x^p e^{-x}\Big|_0^\infty + \int_0^\infty p x^{p-1} e^{-x}\,dx$$

$$= (0-0) + p\int_0^\infty x^{p-1} e^{-x}\,dx$$

$$= 0 + p\Gamma(p)$$

Thus $\boxed{\Gamma(p+1) = p\Gamma(p)}$

26b

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^A = 1$$

26c

$$\Gamma(1)=1 \quad \Gamma(2) = 1\,\Gamma(1)=1 \quad \Gamma(3)=2\,\Gamma(2)=2$$

by induction $\Gamma(n+1)=n!$

26d. Just use part a.

$$\Gamma(p+n) = (p+n-1)\,\Gamma(p+n-1)$$

$$= (p+n-1)(p+n-2)\,\Gamma(p+n-2)$$

$$\vdots$$

$$\boxed{\Gamma(p+n) = (p+n-1)(p+n-2)\cdots(p+1)\,p\,\Gamma(p)}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}/2$$

$\Gamma\left(\frac{11}{2}\right)$

$p=\frac{1}{2}$ $n=\cancel{10}\ 5$

$$\Gamma\left(\tfrac{11}{2}\right) = \tfrac{1}{2}\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)\cdots\left(\tfrac{9}{2}\right)\Gamma\left(\tfrac{1}{2}\right)$$

$$= \boxed{\frac{9\cdot 7\cdot 5\cdot 3}{32}\sqrt{\pi}}$$

27.

$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st} t^p \, dt$

Let $x = st$ $\quad t = \frac{x}{s}$

$dx = s\,dt$

$$\int_0^\infty e^{-st} t^p \, dt = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^p \cdot \frac{1}{s} \, dx$$

$$= \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p \, dx$$

$$= \boxed{\frac{1}{s^{p+1}} \Gamma(p+1)}$$

b. $\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$

c. $\mathcal{L}\{t^{-1/2}\} = \int_0^\infty e^{-st} t^{-1/2} \, dt$

Let $x^2 = st$ $\quad t = \frac{x^2}{s}$

$2x\,dx = s\,dt$

$$= \int_0^\infty e^{-x^2} \left(\frac{x^2}{s}\right)^{-1/2} \frac{2x}{s} \, dx$$

$$= \int_0^\infty e^{-x^2} \frac{\sqrt{s}}{x} \frac{2x}{s} \, dx$$

$$= \boxed{\frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} \, dx}$$

d. ~~Not~~ ...

$\mathcal{L}\{t^{1/2}\} = \frac{\Gamma(3/2)}{s^{3/2}}$ by part a

$= \frac{\sqrt{\pi}}{2s^{3/2}}$ by problem 26

$\Gamma(3/2) = \sqrt{\pi}/2$

p. 322

1. $\frac{3}{s^2+4} = \frac{3}{2}\cdot\frac{2}{s^2+2^2}$ so $\boxed{\frac{3}{2}\sin(2t)}$ by line 5

2. $\frac{4}{(s-1)^3} = 2\cdot\frac{2!}{(s-1)^{3}}$ so $\boxed{2t^2e^t}$ by line 11

6. $\frac{2s-3}{s^2-4} = \frac{2s-3}{(s+2)(s-2)} = \frac{A}{s+2}+\frac{B}{s-2}$

$2s-3 = (A)(s-2)+B(s+2)$

set $s=2$ $\quad 1=4B \quad B=1/4$

$s=-2 \quad -7=-4A \quad A=7/4$

$\frac{7/4}{s+2}+\frac{1/4}{s-2} \Rightarrow \boxed{\frac{7}{4}e^{-2t}+\frac{1}{4}e^{2t}}$

oops, could have used a different idea:

$\frac{2s-3}{s^2-4} = \frac{2s}{s^2-4} - \frac{3}{s^2-4} = 2\cdot\frac{s}{s^2-4} - \frac{3}{s^2-4}$

$\Rightarrow \boxed{2\cosh(2t) - \frac{3}{4}\sinh(2t)}$

Notice these two answers are actually =.

8. $\frac{8s^2-4s+12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

$8s^2-4s+12 = A(s^2+4) + (Bs+C)s$

set $s=0$

$12 = 4A \qquad A=3$

coef of s^2 gives

$8 = A+B \Rightarrow B=5$

coef of s gives $-4 = C$

$\frac{3}{s} + \frac{5s-4}{s^2+4} = \frac{3}{s} + \frac{5s}{s^2+4} - \frac{4}{s^2+4}$

so $\mathcal{L}^{-1}$ gives

$$\boxed{3 + 5\cos(2t) - 2\sin(2t)}$$

21. $y'' - 2y' + 2y = \cos t \quad y(0)=1 \quad y'(0)=0$

$\mathcal{L}(y'') - 2\mathcal{L}(y') + 2\mathcal{L}(y) = \frac{s}{s^2+1}$

$s^2\mathcal{L}(y) - sy(0) - y'(0) - 2(s\mathcal{L}(y) - y(0)) + 2\mathcal{L}(y) = \frac{s}{s^2+1}$

$s^2\mathcal{L}(y) - s - 2s\mathcal{L}(y) + 2 + 2\mathcal{L}(y) = \frac{s}{s^2+1}$

$(s^2-2s+2)\mathcal{L}(y) = \frac{s}{s^2+1} + s-2$

$\mathcal{L}(y) = \frac{s}{(s^2+1)(s^2-2s+2)} + \frac{s-2}{s^2-2s+2} = \frac{s+(s-2)(s^2+1)}{(s^2+1)(s^2-2s+2)}$

21 $y''-2y'+2y=\cos t \quad y(0)=1 \quad y'(0)=0$

$\mathcal{L}(y'')-2\mathcal{L}(y')+2\mathcal{L}(y)=\frac{s}{s^2+1}$

$s^2\mathcal{L}(y)-sy(0)-y'(0)-(2s\mathcal{L}(y)-y(0))+2\mathcal{L}(y)=\frac{s}{s^2+1}$

$(s^2-2s+2)\mathcal{L}(y)-s+1=\frac{s}{s^2+1}$

$\mathcal{L}(y)=\left(\frac{s}{s^2+1}+s-1\right)\cdot\frac{1}{s^2-2s+2}$

$=\frac{s}{(s^2+1)(s^2-2s+2)}+\frac{s-1}{(s-1)^2+1}$

set $\frac{s}{(s^2+1)(s^2-2s+2)}=\frac{as+b}{s^2+1}+\frac{cs+d}{s^2-2s+2}$

$s=(as+b)(s^2-2s+2)+(cs+d)(s^2+1)$

set $s=0$ $\quad 0=2b+d$

s^3 term $\quad 0=a+c$

s^2 term $\quad 0=b-2a+d$

s term $\quad 1=-2b+2a+c$

$c=-a \quad d=-2b$ so $\quad 0=b-2a-2b$

$1=-2b+2a-a$

$=-2b+a$

$0=-b-2a$

$1=a-2b$

$2=-5b$ $\quad b=-2/5$

$a=1+2b=1/5=a$

$c=-1/5$

$d=4/5$

So

$$\mathcal{L}(y) = \frac{1/5\, s}{s^2+1} \quad \frac{-2/5}{s^2+1} + \frac{-1/5\, s + 4/5}{s^2-2s+2} + \frac{s-1}{s^2-2s+2}$$

$$= \frac{1/5\, s}{s^2+1} - \frac{2/5}{s^2+1} + \underbrace{\frac{4/5\, s - 1/5}{s^2-2s+2}}$$

$$= \frac{4/5\, s - 4/5 + 3/5}{(s-1)^2+1}$$

$$= \frac{4}{5} \cdot \frac{s-1}{(s-1)^2+1} + \frac{3}{5} \cdot \frac{1}{(s-1)^2+1}$$

$$\boxed{y = \frac{1}{5}\cos t - \frac{2}{5}\sin t + \frac{4}{5} e^t \cos t + \frac{3}{5} e^t \sin t}$$