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2. $y''+3y'+2y=0$

$r^2+3r+2=0 \qquad (r+2)(r+1)=0$

$$\boxed{y=C_1e^{-2t}+C_2e^{-t}}$$

6. $4y''+0y'-\cancel{9/4}\,y=0$

$4r^2-9=0$

$(2r+3)(2r-3)=0$

$$\boxed{y=C_1e^{-3/2\,t}+C_2e^{3/2\,t}}$$

8. $y''-2y'-2y=0$

$r^2-2r-2=0$

$r=\dfrac{2\pm\sqrt{4+8}}{2}=\dfrac{2\pm2\sqrt{3}}{2}=1\pm\sqrt{3}$

$y=C_1e^{(1+\sqrt{3})t}+C_2e^{(1-\sqrt{3})t}$

10. $y''+4y'+3=0 \qquad y(0)=2 \quad y'(0)=-1$

$r^2+4r+3=0$

$(r+3)(r+1)=0$

$y=C_1e^{-3t}+C_2e^{-t} \qquad 2=C_1+C_2$

$y'=-3C_1e^{-3t}-C_2e^{-t} \qquad -1=-3C_1-C_2$

$1=-2C_1 \qquad C_1=-1/2$

$C_2=5/2$

$$\boxed{y=-\tfrac{1}{2}e^{-3t}+\tfrac{5}{2}e^{-t}}$$

as $t\to\infty$ $y\to0$

15. $y'' + 8y' - 9y = 0 \quad y(1) = 1 \quad y'(1) = 0$

$r^2 + 8r - 9 = 0$

$(r+9)(r-1) = 0$

$y = c_1 e^{t} + c_2 e^{-9t}$

$y' = c_1 e^{t} - 9c_2 e^{-9t}$

$1 = c_1 e + c_2 e^{-9}$

$- \quad 0 = c_1 e - 9c_2 e^{-9}$

$1 = 10c_2 e^{-9}$

$c_2 = \frac{1}{10} e^{9}$

$c_1 e = 9c_2 e^{-9}$

$= 9 \cdot \frac{1}{10} = 9/10$

$c_1 = 9/10 \, e^{-1}$

$y = 9/10 \, e^{-1} e^{t} + \frac{1}{10} e^{9} e^{-9t}$

$$\boxed{y = 9/10 \, e^{t-1} + \frac{1}{10} e^{9-9t}}$$

as $t \to \infty$ the 1st term $\to \infty$

and 2nd $\to 0$

so $y \to \infty$

20. $2y'' - 3y' + y = 0 \quad y(0) = 2 \quad y'(0) = 1/2$

$2r^2 - 3r + 1 = 0$

$(2r-1)(r-1) = 0 \quad r = 1, r = 1/2$

$y = c_1 e^t + c_2 e^{1/2 t}$

$y' = c_1 e^t + \frac{1}{2} c_2 e^{1/2 t}$

$$\begin{array}{r} 2 = c_1 + c_2 \\ -\ 1/2 = c_1 + 1/2\, c_2 \\ \hline 3/2 = 1/2\, c_2 \end{array} \qquad c_2 = 3 \quad c_1 = -1$$

$$\boxed{y = -e^t + 3e^{1/2 t}}$$

to find max value set $y' = 0$.

$y' = -e^t + 3/2\, e^{1/2 t}$

$0 = -e^t + 3/2\, e^{1/2 t}$

$e^t = 3/2\, e^{1/2 t} \qquad e^{1/2 t} = 3/2$

$1/2\, t = \ln(3/2)$

$t = 2\ln(\frac{3}{2}) = \ln(9/4)$

maxvalue is $y(\ln(9/4)) = -e^{\ln(9/4)} + 3e^{\frac{1}{2}(\ln(9/4))}$

(use 1^{st} or 2^{nd} der. test to guarantee local max!

$= 9/4 + 3e^{\ln(3/2)}$

$= -9/4 + 9/2 = 9/4$

Set $y = 0$ $\quad e^t = 3e^{1/2 t}$

$\Rightarrow t = \ln 9$

28. char eq is $ar^2 + br + c = 0$

roots $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

a. Need $\boxed{b^2 - 4ac > 0}$ to make them real and different.

Recall $\frac{-b}{a}$ is the sum of the roots

$\frac{c}{a}$ is the product of the roots

Need $\boxed{b > 0}$ and $\boxed{c > 0}$

to make sum negative

and product positive,

i.e. both roots neg

b. $b^2 - 4ac > 0$ ← automatic since assuming $a > 0$ if $c < 0$

$\boxed{c < 0}$ ← Gives opp signs

c. $\boxed{\begin{array}{c} b^2 - 4ac > 0 \\ b < c \\ c > 0 \end{array}}$

p. 151

2. $\cos t \quad \sin t \qquad W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = \boxed{1}$

4. $W = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x$

$= \boxed{x^2e^x}$

12. $y'' + \frac{1}{x-2} y' + \tan x\, y = 0 \quad y(3) = 1 \; y'(3) = 2$

need $x \neq 2$ and $\tan x$ is not defined at

$x = \pi/2, 3\pi/2, 5\pi/2, \ldots$

Thus $\boxed{(2, 3\pi/2)}$ is the largest interval

containing 3 on which all is continuous

14. $yy'' + (y')^2 = 0$

$y_1 = 1 \quad y_1' = y_1'' = 0 \quad$ so $0 + 0 = 0$.

$y_2 = \sqrt{t} \quad y_2' = \frac{1}{2\sqrt{t}} \quad y_2'' = -\frac{1}{4} t^{-3/2}$

$y_2 y_2'' + (y_2')^2 = -\frac{1}{4} t^{-1} + \frac{1}{4} t^{-1} = 0$

Now let $c_1 + c_2\sqrt{t} = y$

$y' = c_2 \frac{1}{2} t^{-1/2}$

$y'' = -\frac{1}{4} t^{-3/2} c_2$

$yy'' + (y')^2 = -\frac{1}{4} c_1 c_2 t^{-3/2} - c_2^2/4\, t^{-1} + \frac{c_2^2}{4} t^{-1}$

$= -\frac{1}{4} c_1 c_2 t^{-3/2} \neq 0$.

The D.E. is homogeneous but not linear so does not contradict Thm 3.2.2

17. $\begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = g'e^{2t} - 2e^{2t}g$

Given: $g'e^{2t} - 2e^{2t}g = 3e^{4t}$

$g' - 2g = 3e^{2t}$

$(ge^{-2t})' = 3$

$ge^{-2t} = 3t + C$

$g = 3te^{2t} + Ce^{2t}$

22. $y'' + 4y' + 3y = 0$

$r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$y = c_1 e^{-3t} + c_2 e^{-t}$

$y' = -3c_1 e^{-3t} - c_2 e^{-t}$

For Thm 3.2.5 need ~~y(t)~~ $y_1(1) = 1$ $\quad y_2(1) = 0$

$y_1'(1) = 0 \quad y_2'(1) = 1$

$\underline{y_1}$ $\quad 1 = c_1 e^{-3} + c_2 e^{-1}$

$0 = -3c_1 e^{-3} - c_2 e^{-1}$

$-2c_1 e^{-3} = 1$

$c_1 = -\frac{1}{2}e^{3}$

$c_2 = -3c_1 e^{-2}$

$= +\frac{3}{2}e$

$$\boxed{y_1 = -\frac{1}{2}e^{3-3t} + \frac{3}{2}e^{1-t}}$$

To get y_2

$0 = c_1 e^{-3} + c_2 e^{-1}$

$1 = -3c_1 e^{-3} - c_2 e^{-1}$

$1 = -2c_1 e^{-3} \quad c_1 = -\frac{1}{2}e^{+3}$

$c_2 = -c_1 e^{-2} = \frac{1}{2}e^{1}$

$$\boxed{y_2 = -\frac{1}{2}e^{3-3t} + \frac{1}{2}e^{1-t}}$$

25. $x^2y'' - x(x+2)y' + (x+2)y = 0$

$y_1 = x \quad y_1' = 1 \quad y_1'' = 0$

$x^2 \cdot 0 - x(x+2)(1) + x(x+2) = 0$ ✓

$y_2 = xe^x$

$y_2' = xe^x + e^x$

$y_2'' = xe^x + 2e^x$

$x^2 \cdot (xe^x + 2e^x) - x(x+2)(xe^x + e^x) + (x+2)xe^x = 0$ ✓

$W = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x$

$= x^2e^x \neq 0$ for $x > 0$

yes

p.152

28 $P(x)y'' + Q(x)y' + R(x)y = 0$. Suppose this is exact. Then

$P(x)y'' + Q(x)y' + R(x)y = [P(x)y']' + (f(x)y)'$

$= P(x)y'' + P'(x)y' + f'(x)y + f(x)y'$

Setting coefs of y'' & y' & y equal gives:

$P(x) = P(x)$
$Q(x) = P'(x) + f(x)$
$R(x) = f'(x)$

So $f(x) = Q(x) - P'(x)$ $f'(x) = Q'(x) - P''(x)$

$\boxed{R(x) = Q'(x) - P''(x)}$ ← must hold if it is exact

#32 $x^2y'' + xy' - y = 0$ $P = x^2$ $Q = x$ $R = -1$

$Q' - P'' = -1 = R$ so it is exact.

By #28 we see $f(x) = Q(x) - P'(x) = x - 2x = x$

So the D.E. is:

$(x^2y')' + (-xy)' = 0$ integrate:

$x^2y' - xy = C_1$

$y' - \frac{1}{x}y - \frac{C_1}{x^2} = 0$ use Int Fact.

$\mu' = -\frac{1}{x}\mu$ $\frac{\mu'}{\mu} = -\frac{1}{x}$ $\ln|\mu| = -\ln|x| = \ln\frac{1}{x}$

$\mu = \frac{1}{x}$

$(\frac{1}{x}y)' = \frac{C_1}{x^3}$ $\frac{1}{x}y = -\frac{1}{2}C_1x^{-2} + C_2$

$y = -\frac{1}{2}C_1x^{-1} + C_2x$

or ignore $-\frac{1}{2}$ since C_1 is arbitrary

16. $y'' + \tan t\, y' - \frac{t}{\cos t} y = 0$ so $p(t) = \tan t$

$W = ce^{-\int \tan t\, dt}$ $\int \tan t\, dt = \int \frac{-\sin t}{\cos t} = \ln|\cos t|$

$= ce^{\ln|\cos t|} = c\cos t$

17. $y'' + \frac{1}{x}y' + \frac{x^2 - v^2}{x^2} y = 0$

$W = ce^{-\int \frac{1}{x} dx} = ce^{-\ln|x|}$
$= ce^{\ln(\frac{1}{x})} = c \cdot \frac{1}{x}$

19. $(p(t)y')' + q(t)y = 0$

$p(t)y'' + p'(t)y' + q(t)y = 0$

$y'' + \frac{p'(t)}{p(t)}y' + \frac{q(t)}{p(t)}y = 0$

so $W = ce^{-\int \frac{p'(t)}{p(t)} dt}$

$= ce^{-\ln|p(t)|} = ce^{\ln(\frac{1}{p})}$

$= c \cdot \frac{1}{p(t)}$

20. $y'' + \frac{2}{t}y' + e^t y = 0$ $W = ce^{-\int \frac{2}{t} dt} = ce^{-2\ln|t|}$
$= ce^{\ln(\frac{1}{t^2})} = \frac{c}{t^2}$

Given $W(1) = 2$ so $\frac{c}{1^2} = 2$ so $c = 2$ Thus $W = \frac{2}{t^2}$

Now $W(5) = \frac{2}{25}$

24. Suppose $y_1(t_0) = y_2(t_0) = 0$.

If y_1 & y_2 are fundamental solutions then $y = c_1 y_1 + c_2 y_2$

is a general solution, i.e. any solution corresponds to some choice of c_1 & c_2

However If we have initial condition $y(t_0) = 1$ for instance

There is no solution since $y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0)$

$= 0 + 0 = 0$.

Thus they can't be fund. sol.