p. 249

1. $\sum_{n=0}^{\infty} (x-3)^n \qquad \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{(x-3)^n} \right| = |x-3|$

so $|x-3| < 1 \qquad \rho = 1$

5. $\lim_{n\to\infty} \left| \dfrac{\frac{(2x+1)^{n+1}}{(n+1)^2}}{\frac{(2x+1)^n}{n^2}} \right| = \lim_{n\to\infty} |2x+1| \cdot \frac{n^2}{n^2+2n+1}$

$= \lim_{n\to\infty} |2x+1|$

so converges if $|2x+1| < 1$

$-1 < 2x+1 < 1$

$-2 < 2x < 0$

$-1 < x < 0$ so $\rho = 1/2$

7. $\lim_{n\to\infty} \left| \dfrac{\frac{(-1)^{n+1}(n+1)^2(x+2)^{n+1}}{3^{n+1}}}{\frac{(-1)^n n^2 (x+2)^n}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} (x+2) \cdot \frac{1}{3} \right|$

$= \left| \frac{x+2}{3} \right|$

so $\left| \frac{x+2}{3} \right| < 1$

$-3 < x+2 < 3$

$-5 < x < 1 \qquad \rho = 3$

9. $f(0)=0$ $f'(0)=1$ $f''(0)=0$ $f'''(0)=-1$

$f^{(4)}(0)=0$ ...

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right|$$

$= 0 \ \forall x$

so $\rho = \infty$

12. $\boxed{x^2 = (x+1)^2 - 2(x+1) + 1}$ $\rho = \infty$

~~f(x)=x~~ $f(-1)=1$

$f'(-1)=-2$

$f''(-1)=2$

13. $\ln x = f$

$\frac{1}{x} = f'$

$\frac{-1}{x^2} = f''$

$\frac{2}{x^3} = f'''$

$f^n = (-1)^{n+1} \cdot \frac{(n-1)!}{x^n}$

$f(1)=0$

$f'(1)=1$

$f''(1)=-1$

$f'''(1)=2$

$f^n(1)=(-1)^{n+1}(n-1)!$

$\frac{n!}{x^{n+1}}$

$\lim_{n\to\infty} \left| \frac{(n-1)!}{x^n} \right|$

$= \lim$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$

$\lim_{n\to\infty} \left| (x-1) \cdot \frac{n-1}{n} \right|$

$= |x-1|$ so $|x-1|<1$

$\rho = 1$

so $\boxed{\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \dots}$

21. $\displaystyle\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n$

23. $\displaystyle x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{k=0}^{\infty} a_k x^k$

$$= \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \boxed{\sum_{n=0}^{\infty} (n+1)(a_n)\, x^n}$$

p 259

#2. $y'' - xy' - y = 0 \quad x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n \quad y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

So $\displaystyle y'' - xy' - y = \sum_{n=0}^{\infty} \left(-a_n - n a_n + (n+2)(n+1) a_{n+2}\right) x^n$

Thus $\displaystyle a_{n+2} = \frac{(1+n)\, a_n}{(n+2)(n+1)} \qquad \boxed{a_{n+2} = \frac{a_n}{n+2}}$

Suppose $a_0 = 1$ $a_1 = 0$ then $a_2 = \frac{a_0}{2}$ $a_3 = 0$ $a_4 = \frac{a_2}{4} = \frac{a_0}{\cancel{8}4 \cdot 2}$

$a_5 = 0$ $a_6 = \frac{a_4}{\cancel{8}6} = \frac{a_0}{\cancel{64}\, 6 \cdot 4 \cdot 2}$

Notice $10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 2^5 \cdot 5!$ etc...

So $\displaystyle y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = 1 + \frac{x^2}{4} + \frac{x^4}{4 \cdot 2} + \frac{x^6}{2^3 \cdot 6} + \dots$

#2 Suppose $a_0=0$ $a_1=1$

so $a_2=0$ $a_3=\frac{a_1}{3}=1/3$

$a_4=0$, $a_5=\frac{a_3}{5}=\frac{a_1}{5\cdot3}$ $a_6=0$ $a_7=\frac{a_1}{7\cdot5\cdot3}$ etc.

$$y_2(x)=x+\frac{1}{3}x^3+\frac{1}{5\cdot3}x^5+\frac{1}{7\cdot5\cdot3}x^7+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{2^n n!}{(2n+1)!}x^{2n+1}$$

#3 $y''-xy'-y=0$ $x_0=1$

Rewrite as $y''-(x-1)y'-y'-y=0$

$$y=\sum_{n=0}^{\infty}a_n(x-1)^n \quad y'=\sum_{n=1}^{\infty}na_n(x-1)^{n-1} \quad y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}$$

$$(x-1)y'=\sum_{n=1}^{\infty}na_n(x-1)^n=\sum_{n=0}^{\infty}na_n(x-1)^n$$

$$y'=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n$$

$$y''=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n$$

so $y''-(x-1)y'-y'-y=\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}-na_n-(n+1)a_{n+1}-a_n]x^n$

So $(n+2)(n+1)a_{n+2}-(n+1)a_{n+1}=(n+1)a_n$

$$\boxed{a_{n+2}=\frac{a_{n+1}+a_n}{n+2}}$$

#3 (con't) $\cancel{a_0 =}$

Sol 1: $a_0 = 1 \quad a_1 = 0 \quad a_2 = \frac{1}{2} \quad a_3 = \frac{1}{6} \quad a_4 = \frac{1}{6} \ \ldots$

$$y_1 = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 \ldots$$

Sol 2 $a_0 = 0 \quad a_1 = 1 \quad a_2 = \frac{1}{2} \quad a_3 = \frac{1}{2}, \quad a_4 = \frac{1}{4}$

$$y_2 = x - 1 + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4$$

#12 $(1-x)y'' + xy' - y = 0 \quad x_0 = 0$

$y'' - xy'' + xy' - y = 0$

$\cancel{y = \sum_n}$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$\cancel{xy'' =}$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \quad xy'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$$

$$xy'' = \sum_{n=0}^{\infty} (n+1) n a_{n+1} x^n$$

$$xy' = \sum_{n=0}^{\infty} n a_n x^n$$

$$y'' - xy'' + xy' - y = \sum \left[ (n+2)(n+1) a_{n+2} - (n+1) n a_{n+1} + n a_n - a_n \right] x^n$$

So $(n+2)(n+1) a_{n+2} - (n+1) n a_{n+1} + (n-1) a_n = 0$

$$a_{n+2} = \frac{(n+1) n a_{n+1} + (n-1) a_n}{(n+2)(n+1)}$$

#12 cont..

Sol 1 let $a_0=0$ $a_1=1$

$a_2=0$ $a_3=0$ .. so $\boxed{y_1=x}$

Let $a_0=1$ $a_1=0$

$a_2=1/2$ $a_3=1/6$ $a_4=1/24$ ...

$$y=1+\frac{x^2}{2}+\frac{x^3}{6}+\dots+\frac{x^n}{n!}+\dots$$

14. $2y''+(x+1)y'+3y=0$ $x=2$

$$2y''+(x-2)y'+3y'+3y=0$$

$$y=\sum_{n=0}^{\infty}a_n(x-2)^n \quad y'=\sum_{n=1}^{\infty}na_n(x-2)^{n-1} \quad y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-2)^{n-2}$$

$$(x-2)y'=\sum_{n=1}^{\infty}na_n(x-2)^n$$

$$y'=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^n \quad y''=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-2)^n$$

$2y''+(x-2)y'+3y'+3y=$

$$\sum_{n=0}^{\infty}[2a_{n+2}(n+2)(n+1)+na_n+3(n+1)a_{n+1}+3a_n]x^n$$

$$2(n+2)(n+1)a_{n+2}+3(n+1)a_{n+1}+(n+3)a_n=0$$

Let $a_0=1$ $a_1=0$ $y_1(x)=1-\frac{3}{4}(x-2)^2+\frac{3}{8}(x-2)^3+\frac{1}{64}(x-2)^4$

$a_0=0$ $a_1=1$ $y_2(x)=(x-2)-\frac{3}{4}(x-2)^2+\frac{1}{24}(x-2)^3+\frac{9}{64}(x-2)^4+\dots$