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22 $y'' - 2y' + 2y = e^{-t}$ $y(0)=0$ $y'(0)=1$

$s^2\mathcal{L}(y) - sy(0) - y'(0) - 2s\mathcal{L}(y) + 2y(0) + 2\mathcal{L}(y) = \frac{1}{s+1}$

$(s^2-2s+2)\mathcal{L}(y) - 1 = \frac{1}{s+1}$

$\mathcal{L}(y) = \frac{1}{s^2-2s+2} + \frac{1}{(s+1)(s^2-2s+2)}$

$= \frac{1}{(s-1)^2+1} + \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2}$

$1 = A(s^2-2s+2) + (Bs+C)(s+1)$

$s=0$ gives $1 = 2A + C$ $\quad C = 1-2A$

s^2 term $0 = A + B$ $\quad B = -A$

s term $0 = -2A + B + C$ $\quad 0 = -2A - A + 1 - 2A$

$5A = 1$ $\quad A = 1/5 \quad B = -1/5 \quad C = 3/5$

$= -\frac{1}{5}(s-1) + 2/5$

$\mathcal{L}(y) = \frac{1}{(s-1)^2+1} + \frac{1/5}{s+1} \frac{-1/5\, s + 3/5}{(s-1)^2+1}$

$= \frac{1}{(s-1)^2+1} + \frac{1}{5} \cdot \frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{2}{5} \cdot \frac{1}{(s-1)^2+1}$

$= \frac{3}{5} \cdot \frac{1}{(s-1)^2+1} + \frac{1}{5} \cdot \frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1}$

$y = \frac{3}{5} e^t \sin t + \frac{1}{5} e^{-t} - \frac{1}{5} e^t \cos t$

25. $y''+y=\begin{cases} t & 0\le t<1 \\ 0 & 1\le t<\infty \end{cases}$ $y(0)=0$ $y'(0)=0$

Let $g(t)=\begin{cases} t & 0\le t<1 \\ 0 & t\ge 1 \end{cases}$ $\mathcal{L}\{g(t)\}=\int_0^\infty g(t)e^{-st}dt$

$=\int_0^1 te^{-st}dt+0$

$=\frac{1}{s^2}-\frac{e^{-s}}{s^2}-\frac{se^{-s}}{s^2}$

$s^2\mathcal{L}(y)-0-0+\mathcal{L}(y)=\frac{1}{s^2}-\frac{e^{-s}}{s^2}-\frac{se^{-s}}{s^2}$

$\boxed{\mathcal{L}(y)=\frac{1}{s^2}\cdot\frac{1}{s^2+1}(1-e^{-s}-se^{-s})}$

26. $y''+4y=\begin{cases} t & 0\le t<1 \\ 1 & 1\le t<\infty \end{cases}$ $y(0)=y'(0)=0$

$\mathcal{L}\{g(t)\}=\int_0^1 te^{-st}dt+\int_1^\infty e^{-st}dt$

$=\frac{1-e^{-s}-se^{-s}}{s^2}+\lim_{A\to\infty}\frac{e^{-s}}{s}-\frac{e^{-sA}}{s}$

$=\frac{1-e^{-s}-se^{-s}}{s^2}+\frac{e^{-s}}{s}=\frac{1-e^{-s}}{s^2}$

$s^2\mathcal{L}(y)-0-0+4\mathcal{L}(y)=\frac{1-e^{-s}}{s^2}$

$\boxed{\mathcal{L}(y)=\frac{1-e^{-s}}{s^2(s^2+4)}}$

30. $f(t) = \sin bt$ $\quad \mathcal{L}\{f(t)\} = \dfrac{b}{s^2+b^2} = F(s)$

So $\mathcal{L}\{t^2 \sin bt\} = \mathcal{L}\{(-t)^2 f(t)\} = F''(s)$

$F(s) = \dfrac{b}{s^2+b^2}$ $\quad F'(s) = \dfrac{-2bs}{(s^2+b^2)^2}$

$$F''(s) = \boxed{\dfrac{6bs^2-2b^3}{(s^2+b^2)^3}}$$

32. $f(t) = e^{at}$ $\quad \mathcal{L}\{f(t)\} = \dfrac{1}{s-a} = F(s)$

$\mathcal{L}\{t^n e^{at}\} = \mathcal{L}\{(-1)^n(-t)^n e^{at}\}$

$= (-1)^n \mathcal{L}\{(-t)^n e^{at}\}$

$= (-1)^n F^{(n)}(s)$

$F(s) = \dfrac{1}{s-a}$, $\quad -F'(s) = \dfrac{1}{(s-a)^2}$, $\quad +F''(s) = \dfrac{2}{(s-a)^3}$

$$(-1)^n F^{(n)}(s) = \dfrac{n!}{(s-a)^{n+1}}$$

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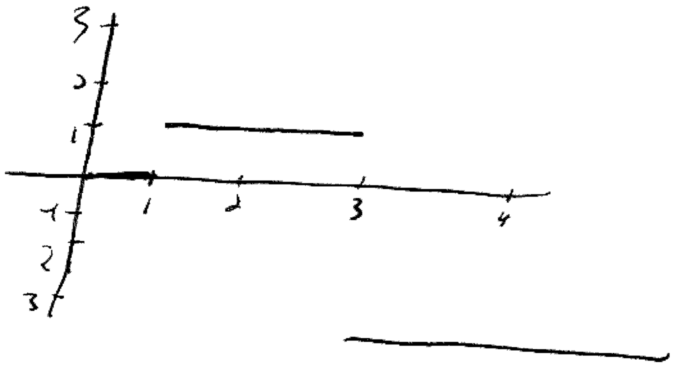
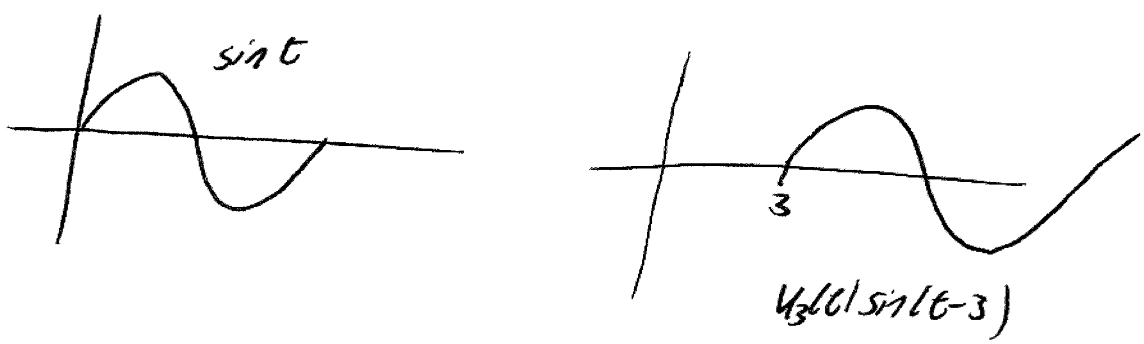
1. $u_1(t)+2u_3(t)-6u_4(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 1 & 1 \le t < 3 \\ 3 & 3 \le t \le 4 \\ -3 & t > 4 \end{cases}$



4.



8. $f(t) = \begin{cases} 0 & t < 1 \\ (t-1)^2 + 1 & t \ge 1 \end{cases}$

Notice $f(t) = u_1(t)\, g(t-1)$ where $g(t) = t^2+1$

Thus $\mathcal{L}\{f\} = e^{-s}\mathcal{L}\{t^2+1\}$

$$= e^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right)$$

16. $F(s) = \dfrac{2e^{-2s}}{s^2-4} = e^{-2s} \cdot \dfrac{2}{s^2-4}$

$$\boxed{u_2(t)\sinh 2(t-2)}$$

17. $F(s) = \dfrac{(s-2)e^{-s}}{s^2-4s+3} = e^{-s}\dfrac{s-2}{(s-3)(s-1)}$

Let $G(s) = \dfrac{s-2}{(s-3)(s-1)} = \dfrac{A}{s-3} + \dfrac{B}{s-1}$

$s-2 = A(s-1) + B(s-3)$

$s=1 \Rightarrow -1 = -2B \qquad B = 1/2$

$s=3 \Rightarrow 1 = 2A \qquad A = 1/2$

$G(s) = \frac{1}{2}\frac{1}{s-3} + \frac{1}{2}\frac{1}{s-1}$

$\mathcal{L}^{-1}\{G\} = \frac{1}{2}e^{3t} + \frac{1}{2}e^{t}$

So by Line 13 $\mathcal{L}^{-1}\{e^{-s}G\} = u_1(t)\left(\frac{1}{2}e^{3(t-1)} + \frac{1}{2}e^{t-1}\right)$

18. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$

$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$ so using line 13

$\mathcal{L}^{-1}\{F\} = u_1(t) + u_2(t) - u_3(t) - u_4(t)$

19. $\mathcal{L}\{f(ct)\} = \int_0^\infty f(ct) e^{-st}\, dt$ let $v = ct$
$dt = \frac{1}{c} dv$

a. $= \int_0^\infty \frac{1}{c} f(v) e^{-\frac{s}{c}v}\, dv$

$= \frac{1}{c} \int_0^\infty e^{-\frac{s}{c}v} f(v)\, dv = \frac{1}{c} F(\frac{s}{c})$

as long as $\frac{s}{c} > a$ i.e. $s > ac$

b. Goal: $\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k} f(\frac{t}{k})$

Just take $\mathcal{L}$ of each side

$\frac{1}{k} \int_0^\infty f(\frac{t}{k}) e^{-st}\, dt$ let $v = \frac{t}{k}$
$dt = k\,dv$

$= \frac{1}{k} \int_0^\infty f(v) e^{-svk} k\, dv$

$= \int_0^\infty e^{-ksv} f(v)\, dv = F(ks)$

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21. $F(s) = \dfrac{2s+1}{4s^2+4s+5} = \dfrac{2s+1}{(2s+1)^2+4}$

so $G(s) = \dfrac{s}{s^2+4}$ $\quad \mathcal{L}^{-1}\{G\} = \cos 2t$

by 14c $\quad \mathcal{L}^{-1}\{F(2s+1)\} = \frac{1}{2} e^{-\frac{t}{2}} \cos \frac{2t}{2}$

$= \boxed{\frac{1}{2} e^{-t/2} \cos t}$

23. $F(s) = \dfrac{e^2 e^{-4s}}{2s-1} = \dfrac{e^{-2(2s-1)}}{2s-1}$

Let $G(s) = \dfrac{e^{-2s}}{s}$

Then $\mathcal{L}^{-1}\{G\} = u_2(t)$ by line 12 / 2

Now $F(s) = G(2s-1)$ so

$\mathcal{L}^{-1}\{F(s)\} = \boxed{\frac{1}{2} e^{+\frac{t}{2}} u_2(t/2)}$

29. $\int_0^2 e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt$

$= -\frac{1}{s}e^{-st}\Big|_0^1 = -\frac{1}{s}(e^{-s}-1) = \frac{1}{s}(1-e^{-s})$

Thus $\mathcal{L}\{f(t)\} = \dfrac{\frac{1}{s}(1-e^{-s})}{1-e^{-2s}} = \dfrac{\frac{1}{s}(1-e^{-s})}{(1-e^{-s})(1+e^{-s})}$

$= \dfrac{\frac{1}{s}}{1+e^{-s}}$

30. $\int_0^2 e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt - \int_1^2 e^{-st}\,dt$

$= \frac{1}{s}(1-e^{-s}) + \frac{1}{s}e^{-st}\Big|_1^2$

$= \frac{1}{s}(1-e^{-s}) + \frac{1}{s}(e^{-2s}-e^{-s})$

So $\mathcal{L}\{f\} = \dfrac{\frac{1}{s}(1-2e^{-s}+e^{-2s})}{\cancel{}\,1-e^{-2s}}$

$u = t \quad v = -\frac{1}{s}e^{-st}$

$du = dt \quad dv = e^{-st}\,dt$

31. $\int_0^1 e^{-st} f(t)\,dt = \int_0^1 te^{-st}\,dt$

$= -\frac{t}{s}e^{-st} + S\,\frac{1}{s}\,e^{-st}\,dt$

$= -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big|_0^1$

$\cancel{= -\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}}$

$\mathcal{L}\{f\} = \dfrac{\left(-\frac{1}{s}-\frac{1}{s^2}\right)e^{-s}+\frac{1}{s^2}}{1-e^{-s}}$