FINAL EXAM SOLUTIONS

1. a. $H \trianglelefteq G$ if $gH = Hg$ $\forall g \in G$.

   b A PID is a ring in which every ideal is principal.

   c. An algebraic # is an element of $\mathbb{C}$ which is a root of a polynomial with rational coefs.

   d $\phi: R \to R'$ is a ring homomorphism if
   $$\phi(x+y) = \phi(x) + \phi(y)$$
   $$\phi(xy) = \phi(x)\phi(y) \qquad \forall x, y \in R.$$

2. a. F b. F c. T d. T e. T f. F

   g. T h. T i. F j. T k. T

3. a. $H \times K = \{(h,k) \mid h \in H, k \in K\}$
   $$(h_1, k_1) \cdot (h_2, k_2) = (h_1 h_2, k_1 k_2)$$

   b $|H| \cdot |K| = |H \times K|$

   c. No, $\mathbb{Z}_2 \times \mathbb{Z}_4$ has no elements of order ≥ 4, so it is not cyclic.

4 a. $p(0) = 1 = p(1)$ so $p(x)$ has no roots so $p(x)$ is irr. since $\deg p \le 3$

b $F = \mathbb{Z}_2[x]/(p(x))$.

Basis is $\{a + b\alpha \mid a, b \in \mathbb{Z}_2,\ \alpha^2 = 1+\alpha\}$

$+$	0	1	α	$1+\alpha$
0	0	1	α	$1+\alpha$
1	1	0	$1+\alpha$	α
α	α	$1+\alpha$	0	1
$1+\alpha$	$1+\alpha$	α	1	0

$\cdot$	0	1	α	$1+\alpha$
0	0	0	0	0
1	0	1	α	$1+\alpha$
α	0	α	$1+\alpha$	1
$1+\alpha$	0	$1+\alpha$	1	α

c. $1 + x + x^2 = (x+\alpha)(x+1+\alpha)$

5. $\sqrt{1+\sqrt{3}}$, $\cos 45°$, $\cos(22.5°)$, $\sqrt{2}$, 27, $\sqrt[4]{2}$, $123/15$ are cons T.

$\cos(20°)$, $\sqrt[3]{2}$ are not

6. $\mathbb{Q}[x,y]/(x^2-y^2)$ is not an integral domain

Proof Let $I = (x^2-y^2)$

$x - y \notin I$, $x+y \notin I$ since I is multiples of x^2-y^2.

Thus $x+y+I \neq 0+I$, $x-y+I \neq 0+I$ but $(x+y+I)(x-y+I) = x^2-y^2+I = 0+I$ so $x+y+I$ and $x-y+I$ are zero divisors in [illegible]

7.

$c=0$ No!

$c=1$ $p(1)=3$ $p(2)=7$ $p(3)=13$ $p(4)=21$ $p(0)=1$

No root $\boxed{c=1}$

$c=2$ $p(1)=4$ $p(2)=8$ $p(3)=14$ $p(4)=22$ $p(0)=2$

$\boxed{c=2}$

$c=3$ $p(1)=0$ No!

$c=4$ $p(1)=6$ $p(2)=10$ No!

8. $\{(0,n) \mid n \in \mathbb{Z}\}$ is prime but not maximal!

9. Let a,b be nilpotent, $a,b \in n(R)$.

Thus $a^n=0$, $b^m=0$ for some $n,m \in \mathbb{Z}$

$$(a+b)^{n+m} = a^{n+m} + a^{n+m-1}b + \dots = 0+0+\dots+0 = 0$$

So $a+b \in n(R)$. Let $r \in R$

$(ar)^n = a^n r^n = 0$ since R is commutative

$\overset{\shortparallel}{(ra)^n}$ So $ar, ra \in n(R)$

Thus $n(R)$ is an ideal!

9.

Claim: $\mathfrak{n}(R/\mathfrak{n}(R))=0$

Pf. Suppose $a+\mathfrak{n}(R)$ is nilpotent, so

$(a+\mathfrak{n}(R))^n=0$ some n.

Thus $0=(a+\mathfrak{n}(R))^n=a^n+\mathfrak{n}(R)=0+\mathfrak{n}(R)$. so

$a^n\in\mathfrak{n}(R)$, i.e. a^n is nilpotent. Thus

$(a^n)^m=0$ for some m so $a^{nm}=0$ so $a\in\mathfrak{n}(R)$

so $a+\mathfrak{n}(R)=0+\mathfrak{n}(R)$. Thus $R/\mathfrak{n}(R)$ has

no nonzero nilpotent elements.

10. a. 1, 5, 7, 11, 13, 17

b.

$$\begin{tikzcd}
 & \mathbb{Z}_{18}=\langle 1\rangle \arrow[dl, dash] \arrow[dr, dash] & & \\
\mathbb{Z}_9=\langle 2\rangle \arrow[dr, dash] & & \mathbb{Z}_6=\langle 3\rangle \arrow[dl, dash] \arrow[dr, dash] & \\
 & \mathbb{Z}_3=\langle 6\rangle \arrow[dr, dash] & & \mathbb{Z}_2=\langle 9\rangle \arrow[dl, dash] \\
 & & 0 &
\end{tikzcd}$$

$x, y \in H \cap K$. Then

$xy^{-1} \in H$ since $x, y \in H$ and $H \leq G$.
$xy^{-1} \in K$ since $x, y \in K$ and $K \leq G$.

Thus $xy^{-1} \in H \cap K$ so $H \cap K$ is a subgroup.

b Let $G = \mathbb{Z}$ $H = 3\mathbb{Z}$ $K = 5\mathbb{Z}$

$H \cup K$ = all multiples of 3 and/or 5 is ~~not~~ a su

12. Let $\phi(n) \in \Phi[N]$ and $\phi(g) \in \Phi[G]$

$\phi(g)\,\phi(n)\,\phi(g)^{-1} = \phi(gng^{-1})$ since ϕ is a hom

$= \phi(n')$ $n' \in N$ since $N \triangleleft$

$\in \phi[N]$.

Thus $\phi[N] \triangleleft \phi[G]$.