Math 3320- Final Exam - May 6, 2005

1. **(12 points)** Define the following terms:

a. normal subgroup
b. integral domain
c. algebraic number
d. ring homomorphism

2. **(22 points)** True or false:

____ a. If R is an integral domain then any quotient ring R/N is an integral domain.

____ b. If R is a commutative ring and R/N is an integral domain then R must be an integral domain.

____ c. If D is an integral domain then $D[x]$ is an integral domain.

____ d. If R is a ring containing zero divisors then $R[x]$ contains zero divisors.

____ e. Addition in every ring is commutative.

____ f. The nonzero elements in a ring form a group under multiplication.

____ g. It makes sense to speak of the factor group G/N if and only if N is a normal subgroup of the group G.

____ h. Every finite group is isomorphic to a subgroup of some symmetric group.

____ i. Any two groups of 4 elements are isomorphic.

____ j. Any two groups of 7 elements are isomorphic.

____ k. Every infinite cyclic group is isomorphic to the integers under addition.

3. **(16 points)**

a. Let H and K be groups. Explain how to construct the direct product group $H \times K$. That is, explain what the elements of the direct product are and how the group operation is defined.

b. If H and K are finite groups, how many elements does the group $H \times K$ have?

c. Is $\mathbf{Z}_2 \times \mathbf{Z}_4 \cong \mathbf{Z}_8$? Explain your answer.

4. **(15 points)** Consider the polynomial $p(x) = x^2 + x + 1 \in \mathbf{Z}_2[x]$.

a. Show that $p(x)$ is irreducible over $\mathbf{Z}_2$.

b. Construct an extension field of $\mathbf{Z}_2$ which contains a root of $p(x)$. Give the complete multiplication and addition table for the field.

c. Factor $p(x)$ into two degree one factors over the extension field.

5. **(15 points)** Recall that a real number α is *constructible* if we can construct a line segment of length $|\alpha|$ in a finite number of steps from a segment of unit length using only a straightedge and compass. Which of the ten real numbers below are constructible:

$$\pi,\ \cos(20^o),\ \sqrt{1+\sqrt{3}},\ \cos(45^o),\ \sqrt[3]{2},\ \cos(22.5^o), \sqrt{2},\ 27,\ \sqrt[4]{2},\ \frac{123}{15}$$

6. **(15 points)** Let $\mathbf{Q}[x, y]$ be the ring of polynomials with rational coefficient in two variables x and y. Let $N = (x^2 - y^2)$, i.e. N is the principal ideal generated by $x^2 - y^2$. Is the quotient ring $\mathbf{Q}[x, y]/N$ an integral domain? If so prove it. If not, demonstrate why not.

7. **(10 points)** Find all $c \in \mathbf{Z}_5$ such that $\mathbf{Z}_5[x]/(x^2 + x + c)$ is a field.

8. **(10 points)** Find a prime ideal of $\mathbf{Z} \times \mathbf{Z}$ that is not maximal.

9. **(20 points)** Let R be a commutative ring. An element $x \in R$ is called *nilpotent* if $x^n = 0$ for some $n \in \mathbf{Z}$.

a. Prove that the set of nilpotent elements forms an ideal, called the *nilradical* and denoted $\eta(R)$.

b. Prove that $\eta(R/\eta(R)) = 0$. (I.e. show that the quotient ring $R/\eta(R)$ has no nonzero nilpotent elements.)

10. **(15 points)**

a. What are the generators for the cyclic group Z_{18} under $+_{18}$?

b. Give the complete subgroup lattice of Z_{18}.

11. **(15 points)**

a. Let $H \leq G$ and $K \leq G$. Prove that $H \cap K \leq G$.

b. Give an example to show that $H \cup K$ need not be a subgroup of G.

12. **(15 points)** Let $\phi : G \to G'$ be a group homomorphism. Let $N \trianglelefteq G$. Prove $\phi[N] \trianglelefteq \phi[G]$.

13. **(20 points)** Consider the two subgroups of S_4:

$$V = \{e,\ (12)(34),\ (13)(24),\ (14)(23)\}.$$

$$H = \{e,\ (12)\,, (34)\,, (12)(34)\}$$

Notice that $V \cong H$. The cosets of V are written below for your convenience. Prove that V is normal in S_4 and H is not normal in S_4. Construct the multiplication table for the quotient group S_4/V. Is the quotient group cyclic?

$$\begin{array}{rcllcl}
V & = & \{e,\ (12)(34),\ (13)(24),\ (14)(23\} & & & \\
(12)V & = & \{(12),\ (34),\ (1324),\ (1423)\} & & = & V(12) \\
(13)V & = & \{(13),\ (1234),\ (24),\ (1432)\} & & = & V(13) \\
(14)V & = & \{(14),\ (1243),\ (1342),\ (23)\} & & = & V(14) \\
(123)V & = & \{(123),\ (134),\ (243),\ (142)\} & & = & V(123) \\
(124)V & = & \{(124),\ (143),\ (132),\ (234)\} & & = & V(124)
\end{array}$$