

Math 3320 Midterm Exam #3 - April 14, 2005

1. (15 points) Define the following terms:

- a. integral domain
- b. zero divisor
- c. characteristic of a ring R
- d. ideal
- e. ring homomorphism

2. (16 points) True or false:

- _____ a. If R is an integral domain then any quotient ring R/N is an integral domain.
- _____ b. If R is a commutative ring and R/N is an integral domain then R must be an integral domain.
- _____ c. The set of polynomials of degree ≥ 2 forms an ideal in the ring $\mathbf{Z}[x]$.
- _____ d. The set of polynomials of *even* degree ≥ 2 forms an ideal in the ring $\mathbf{Z}[x]$.
- _____ e. The set of units in a ring R is closed under addition.
- _____ f. The characteristic of a subring must be the same as the characteristic of the ring.
- _____ g. \mathbf{R} is the field of quotients of \mathbf{Z} .
- _____ h. A degree three polynomial in $\mathbf{Z}_{20}[x]$ has at most three roots in \mathbf{Z}_{20} .

3. (15 points) Let R be a commutative ring with ideal N . Define the *radical* of N to be:

$$\sqrt{N} = \{a \in R \mid a^n \in N \text{ for some } n \in \mathbf{Z}^+\}.$$

That is, the radical is all elements for which some power of the element is in N .

- a. Check that $N \subseteq \sqrt{N}$.
- b. Prove that \sqrt{N} is an ideal of R .
- c. Prove that $\sqrt{\sqrt{N}} = \sqrt{N}$.
- d. Let $R = \mathbf{Z}$ and $N = 8\mathbf{Z}$. Determine \sqrt{N} .

4. (10 points) Let $\phi : R \rightarrow R'$ be a ring homomorphism. Prove that $\ker\phi$ is an ideal of R .

5. (12 points) State the Eisenstein criterion for a polynomial in $\mathbf{Z}[x]$ to be irreducible over \mathbf{Q} . Then for each polynomial below either use the criterion to prove the polynomial is irreducible $\mathbf{Q}[x]$ or state that the criterion does not apply:

$$x^3 + 6x^2 - 12$$

$$6x^4 - 3x^2 - 9x + 3$$

$$5x^2 + 10x - 6$$

6. (12 points) Let ϕ denote the Euler phi-function and let p, q be distinct primes.

a. Determine $\phi(p^2)$ and $\phi(pq)$

b. State Euler's generalization of Fermat's Little Theorem.

c. Then use it to calculate the remainder of 37^{50} when it is divided by 8.

7. (8 points) Give an example of each:

a. A integral domain that is not a field.

b. A commutative ring without identity.

c. A noncommutative ring with identity.

d. A commutative ring with characteristic 3 that is not a field.

8. (12 points) Let R be a ring with identity and $U(R)$ the set of units in R .

a. Prove that $U(R)$ is a group under multiplication.

b. For each of the following rings give the units:

$$\begin{array}{c} \mathbf{Z} \\ \mathbf{Q} \\ \mathbf{Z}/n\mathbf{Z} \\ \mathbf{Z}[x] \\ M_n(\mathbf{R}) \text{ (} n \times n \text{ matrices with real entries).} \end{array}$$