Exam 2 Solutions

1\. a. the # of left cosets which is equal to $\frac{|G|}{|H|}$ if groups are finite.

b. $\ker \phi = \{ g \in G \mid \phi(g) = e \}$

c. $H \trianglelefteq G$ if $gH = Hg \quad \forall g \in G$

d. G is simple if G and $\{e\}$ are the only normal subgroups

2\. a. T b. T c. F d. T e. F f. F g. T

3\.
a. $H \times K = \{ (h,k) \mid h \in H, k \in K \}$

The operation is coordinatewise:

$$(h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 k_2)$$

b. $|H| \cdot |K|$

c.

	order
$(0,0)$	1
$(1,0)$	2
$(0,1)$	3
$(1,1)$	6
$(0,2)$	3
$(1,2)$	6

yes it is cyclic, $\cong$ to $\mathbb{Z}_6$

4.

$\sigma\tau = (123)(45)(1342) = \boxed{(354)}$

$\tau\sigma^2 = (1342)(123)(45)(123)(45) = \boxed{(1423)}$

$\tau\sigma\tau^{-1} = (1342)(123)(45)(1243) = \boxed{(143)(25)}$

$\sigma^{38}\tau^{101} = \sigma^2\tau$ since $\sigma^6 = e$, $\tau^4 = e$

$= (123)(45)(123)(45)(1342) = \boxed{(1234)}$

5.

a. Let $h_1 n_1$ and $h_2 n_2$ be $\in HN$.

$$h_1 n_1 h_2 n_2 = \cancel{h_1 h_1}^{-1} h_1 h_2 h_2^{-1} n_1 h_2 n_2$$

$$= (h_1 h_2)(h_2^{-1} n_1 h_2) n_2$$

but $h_2^{-1} n_1 h_2 \in N$ since $N \triangleleft G$.

Thus $(h_1 h_2)(h_2^{-1} n_1 h_2 n_2) \in HN$ so HN is closed under $\cdot$.

$$(h_1 n_1)^{-1} = n_1^{-1} h_1^{-1} = h_1^{-1} \underbrace{h_1 n_1^{-1} h_1^{-1}}_{\in N \text{ since } N \triangleleft G}$$

$\in HN.$

Thus $(h_1 n_1)^{-1} \in HN$ so $HN \leq G$.

b. We must show $ghng^{-1} \in HN \quad \forall g \in G$

$$ghng^{-1} = \underbrace{ghg^{-1}}_{\substack{\in H \\ \text{since } H \trianglelefteq G}} \underbrace{gng^{-1}}_{\in N \text{ since } N \trianglelefteq G}$$

So $ghng^{-1} \in HN$ so $HN \trianglelefteq G$.

6.

Let $x, y \in G$.

$$\begin{aligned}(\nu\phi)(xy) &= \nu(\phi(xy)) \\ &= \nu(\phi(x)\phi(y)) \quad \text{since } \phi \text{ is a homom.} \\ &= \nu(\phi(x))\nu(\phi(y)) \quad \text{since } \nu \text{ is a homom.} \\ &= (\nu\phi)(x)\,(\nu\phi)(y)\end{aligned}$$

Thus $\nu\phi$ is a homom.

6'. Suppose ϕ is 1-1. $\phi(e) = e$ so $\phi(x)$ cannot be e for any $x \neq e$, thus $\ker\phi = \{e\}$

Conversely assume $\ker\phi = \{e\}$. Suppose $\phi(x) = \phi(y)$. we must show $x = y$.

But $\phi(x) = \phi(y) \Rightarrow \phi(x)\phi(y^{-1}) = e$

$\phi(xy^{-1}) = e$

$xy^{-1} \in \ker\phi$

So $xy^{-1} = e$

So $x = y$. Thus ϕ is 1-1

7.

a.

$$Z = \{1, -1\}$$
$$iZ = \{i, -i\} = Zi$$
$$jZ = \{j, -j\} = Zj$$
$$kZ = \{k, -k\} = Zk$$

so $Z \trianglelefteq Q$.

b.

	Z	iZ	jZ	kZ
Z	Z	iZ	jZ	kZ
iZ	iZ	Z	kZ	jZ
jZ	jZ	kZ	Z	iZ
kZ	kZ	jZ	iZ	Z

c. V

8. a. G is abelian if & only if $xy = yx$ $\forall x, y \in G$.
if & only if $x^{-1}y^{-1}xy = e$ $\forall x, y \in G$.

b. G/H is abelian if & only if $aHbH = bHaH$ $\forall aH, bH \in G/H$
if & only if $abH = baH$.

But by the criterion for equality of left cosets this holds if & only if $a^{-1}b^{-1}ab \in H$ $\forall a, b \in H$.