Exam 1 Solutions

1. a. A map ϕ such that $\phi(xy) = \phi(x)\phi(y)$ $\forall x, y \in G$.

   b. A homomorphism which is 1-1 and onto

   c. The least integer $n > 0$ such that $g^n = e$.

   d. Permutations of $\{1, 2, \ldots, n\}$

   e. $\langle g \rangle = \{ g^n \mid n \in \mathbb{Z} \}$

2. a. T b F c. F d. T e. F f. F g. F h. T

3. a. yes b No inverses c. yes d. Not closed under +

   e. No, 4 has no inverse f. yes g. yes h. No identity

4. a. $geg^{-1} = e$ $\forall g$ so e is only conjugate to itself.

   b. If G is abelian then $gxg^{-1} = gg^{-1}x = x$ so x is only conjugate to itself.

   c. Need $g \sim g$ $\forall g \in G$.

   $g \sim h \Rightarrow h \sim g$

   $g \sim h$ and $h \sim k \Rightarrow g \sim k$

4 d.

i) $x = exe^{-1}$ so $x \sim x$.

ii) Suppose $x \sim y$ so $gxg^{-1} = y$ for some g. Then

$$y = g^{-1}xg = g^{-1}x(g^{-1})^{-1} \text{ so } y \sim x.$$

iii) Suppose $x \sim y$ and $y \sim z$. Then $\exists\, g_1, g_2 \in G$ so

$$g_1 x g_1^{-1} = y \quad \text{and} \quad g_2 y g_2^{-1} = z.$$

Thus
$$\begin{aligned} z &= g_2 y g_2^{-1} \\ &= g_2 g_1 x g_1^{-1} g_2^{-1} = (g_2 g_1) x (g_2 g_1)^{-1} \end{aligned}$$

so $x \sim z$.

5. a. $\{1, 5, 7, 11, 13, 17\}$

b.
$$\begin{tikzcd}
 & \mathbb{Z}_{18} \arrow[dl, dash] \arrow[dr, dash] & & \\
\langle 2 \rangle \arrow[dr, dash] & & \langle 3 \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
 & \langle 6 \rangle \arrow[dr, dash] & & \langle 9 \rangle \arrow[dl, dash] \\
 & & \{0\} &
\end{tikzcd}$$

6. a. Let $g_1, g_2 \in K$. So $\phi(g_1) = \phi(g_2) = e$.

$\phi(g_1 g_2) = \phi(g_1)\phi(g_2) = ee = e$. Thus $g_1 g_2 \in K$.

$\phi(g_1^{-1}) = \phi(g_1)^{-1} = e^{-1} = e$ so $g_1^{-1} \in K$.

Thus $K \leq G$.

b. The identity in $\mathbb{R}^+$ is 1. Thus

$K = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle,

which is a subgroup

Notice $|z_1 z_2| = |z_1||z_2|$ so ϕ is a homom.

7. a. No

b. $\sigma(-1) = 2$ $\sigma(i) = \sigma(-k) = 4$

c. $-i$

d. $\{1, -1, k, -k\}$

e. This group has only two elements which square to e. D_4 has at least 4 (corresponding to the "flips")