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#32 a. T b. T c. F d. T e. F

f. F g. T h. T i. F j. F

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4. 3

6. $|\langle(4,3)\rangle|$ is 6 so

$$|\mathbb{Z}_6 \times \mathbb{Z}_{18}| / \langle(4,3)\rangle = \frac{12 \cdot 18}{6} = 36$$

14. $(3,3) + (3,3) = (2,6) \in \langle(1,2)\rangle$

$3 \cdot (3,3) = (1,1)$ ~~so order is 2.~~

$4 \cdot (3,3) = (0,4)$

$(3,0), (2,3), (4,5)$ $(4,2)$ $(2,4)$

order $= 8$

order $8 \cdot (3,3) = (4,0)$ is smallest mult in $\langle(1,2)\rangle$

15. 1 since $(2,0) \in \langle(4,4)\rangle$ then $(2,0) + \langle(4,4)\rangle = \langle(4,4)\rangle$

22. Thm G is torsion $\Rightarrow$ G/H is torsion.

Proof Let $xH \in G/H$. G is torsion so x has finite order n. Thus $x^n = e$. So

$$(xH)^n = x^n H = eH = H.$$

Hence the order of xH in G/H is $\leq n$ so G/H is torsion.

23. a. T b. T c. T d. T e. T

f. F g. T h. F i. T j. F

26. Any subgroup of an abelian group is normal so certainly $T \triangleleft G$.

<u>Claim</u> G/T is torsion free

<u>Proof</u> We must prove that an element of G/T which has finite order is actually the identity.

Suppose $(gT)^n = e$, i.e. $(gT)^n = T$, so $g^nT = T$.

Thus $g^n \in T$. but elements of T have finite order so g^n has finite order k, i.e. $g^{nk} = e$. Thus g has finite order so $g \in T$ so $gT = T$.

Thus G/T is torsion free.

27. a $H = eHe^{-1}$ so $H \sim H$

b If $K = gHg^{-1}$ then $H = g^{-1}Kg = i_{g^{-1}}(K)$

Thus if $H \sim K$ then $K \sim H$.

c. Suppose $H \sim K$ & $K \sim L$. Then

$K = gHg^{-1}$ and $L = g'Kg'^{-1}$ for some $g, g' \in G$.

Thus $L = g'gHg^{-1}g'^{-1} = (g'g)H(g'g)^{-1}$ so $H \sim L$.

Thus conjugacy is an equiv. relation.

28. Normal subgroups are those such that $i_g(H) = H \; \forall g$

i.e. cells with one element.

~~30~~ 29. $H = \{e, (13)\}$

$(12)H(12)^{-1} = \{e, (23)\} = (132)H(132)^{-1}$

sorry, not assigned

$(23)H(23)^{-1} = \{e, (12)\} = (123)H(123)^{-1}$

32. The smallest normal subgroup containing S is the intersection of all normal subgroups containing S.

This gives a normal subgroup by #31.

Obviously this intersection sits inside any normal subgroup containing S, so it is the smallest.

34. See exercise 27. gHg^{-1} is a subgroup of G with the same # of elements as H. But if H is the only such subgroup then

$$gHg^{-1} = H \quad \forall g \in G$$

$$\text{i.e. } H \lhd G.$$

30. $H \lhd G$ $m = [G:H]$

Since $m = |G/H|$ we know $(aH)^m = H \quad \forall aH \in G/H.$

$$\text{Thus } a^m H = H \ \forall a \in G$$

$$\text{so } a^m \in H \ \forall a \in G.$$

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34. If G has a subgroup of index 2 it must be normal so G cannot be simple.

35. Suppose $N \lhd G$. Show $\phi(N) \lhd G'$

Proof Let $\phi(n) \in \phi(N)$ and $\phi(g) \in \phi[G]$.

We must prove $\phi(g)\,\phi(n)\,\phi(g)^{-1} \in \phi(N)$ to show $\phi(N) \lhd \phi(G)$!
But:

$$\phi(g)\,\phi(n)\,\phi(g^{-1}) = \phi(gng^{-1})$$

but $N \lhd G$ so $gng^{-1} \in N$

so $\phi(gng^{-1}) \in \phi(N)$ as desired!

36. Let $\phi: G \to G'$, $N' \lhd G'$ Show $\phi^{-1}(N') \lhd G$.

Proof Let $h \in \phi^{-1}(N')$, so $\phi(h) \in N'$. Let $g \in G$.
We must prove

$$gh g^{-1} \in \phi^{-1}(N') \quad \text{But}$$

$$\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g)^{-1}$$

$\in N'$ since $\phi(h) \in N'$ and $N' \lhd G'$.

Thus $ghg^{-1} \in \phi^{-1}(N')$

40. Given $N \trianglelefteq G$ $H \leq G$. Let $HN = \{hn \mid h \in H, n \in N\}$.
Show $HN \leq G$.

Proof Let $h_1 n_1$ and $h_2 n_2$ be $\in HN$. Then

$$h_1 n_1 h_2 n_2 = h_1 h_2 h_2^{-1} n h_2 n_2$$

but $N \trianglelefteq G$ so $h_2^{-1} n h_2 = n' \in N$.

$$= h_1 h_2 n' n_2 \in HN.$$

So HN is closed under $\bullet$.

$$(h_1 n_1)^{-1} = n_1^{-1} h_1^{-1} = h_1^{-1} h_1 n_1^{-1} h_1^{-1}$$

$$= h_1^{-1} n'' \quad \text{since } h_1 n_1^{-1} h_1^{-1} \in N$$

$$\in HN$$

so HN is closed under inverses
Thus $HN \leq G$. //

Notice if $G' \leq G$ is a subgroup containing H & N then it must contain all hn since it is closed under $\bullet$.
Thus

$HN \subseteq$ any subgroup containing H & N.

Since HN is actually a subgroup it makes sense to call it the smallest subgroup containing H & N.

42.

$$hkh^{-1}k^{-1} = \underbrace{(hkh^{-1})}_{\text{in } K}k^{-1} \in K$$

$$hkh^{-1}k^{-1} = h\underbrace{(kh^{-1}k^{-1})}_{\text{in } H} \in H.$$

Thus $hkh^{-1}k^{-1} \in H \cap K = \{e\}$

so $hkh^{-1}k^{-1} = e$

so $hk = kh$.