2. $(a*b)*c = (b*c) = a$
   $a*(b*c) = a*a = a$ No way! You must check all possible 3 way multiplications.

4. yes! You can test this by noticing the table is symmetric about the ($\diagdown$) diagonal!

6. Notice $d = c*b$. Using this and associativity we can determine $d*a$, $d*b$, $d*c$ & $d*d$

$$d*a = (c*b)*a = c*(b*a) = c*b = d$$
$$d*b = (c*b)*b = c*(b*b) = c*a = c$$
$$d*c = (c*b)*c = c*(b*c) = c*c = c$$
$$d*d = (c*b)*(c*b) = c*(b*c)*b$$
$$= c*c*b = c*b = d$$

8. obviously commutative

$$a*(b*c) = \cancel{a*(} \; a\cdot(b*c)+1$$
$$= a(bc+1)+1 = abc+a+1$$
$$(a*b)*c = \cancel{(ab+} \cancel{(a}$$
$$(a*b)c+1 = (ab+1)c+1$$
$$= abc+c+1$$

Not associative.

11. obviously not commutative, $a^b \neq b^a$

$$(a*b)*c = (a*b)^c = (a^b)^c = a^{bc}$$

$$a*(b*c) = a^{b*c} = a^{(b^c)} = a^{(b^c)}$$

not associative.

23 a. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix}$

closed under +

b $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{pmatrix} ac-bd & -ad-bc \\ bc+ad & ac-bd \end{pmatrix}$

$= \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}$ closed under •

24 a. F b. T c. F d. F e. F
f. T g. F h. F i. T j. F

36. Let $h_1, h_2 \in S$ Then $h_1 * x = x * h_1 \quad \forall x \in S$
$h_2 * x = x * h_2 \quad \forall x \in S$

So $(h_1 * h_2) * x = h_1 * (h_2 * x)$ by assoc. law
$= h_1 * (x * h_2)$ since $h_2 \in H$
$= (h_1 * x) * h_2$ by assoc. law
$= (x * h_1) * h_2$ since $h_1 \in H$
$= x * (h_1 * h_2)$ by ass. law $\quad \forall x \in S$

Thus $h_1 * h_2 \in H$, i.e. H is closed under $*$.

37. Let $h_1, h_2 \in H$, so $h_1 * h_1 = h_1$, $h_2 * h_2 = h_2$.

Then $(h_1 * h_2) * (h_1 * h_2) = h_1 * h_1 * h_2 * h_2$ using assoc. law and commutativity

$= h_1 * h_2$

Thus $h_1 * h_2 \in H$.

page 34

4. No. $\phi(n_1 + n_2) = n_1 + n_2 + 1$

$\phi(n_1) + \phi(n_2) = n_1 + 1 + n_2 + 1 = n_1 + n_2 + 2$

6. No. ϕ is not 1-1 since $\phi(x) = \phi(-x)$

Also ϕ is not onto since only rationals ≥ 0 are in the range.

7. Yes! ϕ is 1-1 and onto.

Also $\phi(x_1 x_2) = (x_1 x_2)^3 = x_1^3 x_2^3$

$= \phi(x_1)\phi(x_2)$

10. yes! $\phi(r_1 + r_2) = 0.5^{r_1 + r_2} = .5^{r_1} .5^{r_2}$

$= \phi(r_1) \cdot \phi(r_2)$

16.

a. Let x and $y \in \mathbb{Z}$ Notice $x = \phi(x-1)$

$y = \phi(y-1)$

To define $x * y$ we know

$$x * y = \phi(x-1) * \phi(y-1)$$
$$= \phi((x-1)(y-1)) = (x-1)(y-1)+1$$

$$\boxed{x * y = (x-1)(y-1) + 1}$$

b. $\phi(x * y) = \phi(x) + \phi(y)$
$= x+1+y+1$
$= x+y+2$

So $x * y$ must map to $x+y+2$ under ϕ

Hence

$$x * y = x+y+1$$

29. Thm Suppose $\langle S, * \rangle$ is commutative and $\phi: S \to S'$ is an isomorphism with $\langle S', *' \rangle$. Then $*'$ is commutative.

Proof Let $s_1', s_2' \in S'$ Then ϕ is 1-1 & onto so choose

$s_1 \in S$, $s_2 \in S$ such that

$$\phi(s_1) = s_1' \quad \phi(s_2) = s_2'$$

29 (cont)

Then $s_1' *' s_2' = \phi(s_1) *' \phi(s_2)$

$\quad '' \quad = \phi(s_1 * s_2)$ since ϕ is a homo

$\quad '' \quad = \phi(s_2 * s_1)$ since S is commutative

$\quad '' \quad = \phi(s_2) *' \phi(s_1)$ since ϕ is a homo

$s_1' *' s_2' = s_2' *' s_1'$

Thus $(S', *')$ is commutative.

30. Let ϕ, S, S' be as in #29

Thm If $\exists\, b$ in S with $b * b = b$ Then same holds for S'.

Proof Suppose $b * b = b$

$\phi(b) = \phi(b * b)$

$\quad = \phi(b) *' \phi(b)$

so $\phi(b)$ has the property