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#3 max: $\mathbb{Z}_2 \times \{0\}$ or $\{0\} \times \mathbb{Z}_2$
or
prime

5. Need $x^2 + c$ irred in $\mathbb{Z}_3$ $\boxed{c=1}$ since $x^2+2 = (x+2)(x+1)$

6. $c = 2$

15. max ideals of $\mathbb{Z} \times \mathbb{Z}$ are of the form

$$p\mathbb{Z} \times \mathbb{Z} \quad \text{or} \quad \mathbb{Z} \times p\mathbb{Z}$$

18. $x^2 - 5x + 6 = (x-3)(x-2)$ so $\mathbb{Q}[x] / \langle x^2 - 5x + 6 \rangle$ is not a field

19. $x^2 - 6x + 6$ is irr.
over $\mathbb{Q}$ by E.C. w/ $p=2$ or $p=3$
so $\boxed{\text{yes}}$

24. Let N be a prime ideal. Then
R/N is a <u>finite</u> integral domain,
so R/N is a field (proven in class)!
Thus N is maximal.

30. Let N be a proper nontrivial prime ideal of $F[x]$

By Thm 27.24 $N = (p(x))$. If $p(x) = r(x)\,q(x)$ with degrees > 0

Then N is not prime since $p(x) \in N$, $r(x), q(x) \notin N$

Thus $p(x)$ is irreducible so N is maximal by Thm 27.25.

34. Let $x = a_1 + b_1$, $y = a_2 + b_2 \in A+B$, Let $r \in R$

a. $x+y = (a_1+a_2) + (b_1+b_2) \in A+B$

$ra_1 \in A$, $rb_1 \in B$ since A, B are ideals
$a_1 r \quad b_1 r$

Thus $r(a_1+b_1) = ra_1 + rb_1 \in A+B$

$(a_1+b_1)\,r = a_1 r + b_1 r \in A+B$

Thus $A+B$ is an ideal

b. $a = a+0 \in A+B$ for $a \in A$ so $A \subseteq A+B$

$b = 0+b \in A+B$ for $b \in B$ so $B \subseteq A+B$

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Let $x = \sum_{i=1}^{n} a_i b_i$, $y = \sum_{j=1}^{m} a_j' b_j'$

Obviously x & y are of the same form so $\in AB$

$rx = \sum r a_i b_i$ but $r a_i \in A$ so $rx \in AB$
$xr = \sum a_i b_i r$ but $b_i r \in B$ so $xr \in AB$

Thus AB is an ideal

But $a_i b_i \in A$ since A is an ideal
$a_i b_i \in B$ since B is an ideal

Thus $AB \subseteq A$
$AB \subseteq B$ $\Rightarrow AB \subseteq A \cap B$

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30. By Thm 29.18 every element of $F(\alpha)$ has a unique expression of the form

$$a_0 + a_1\alpha + \dots + a_{n-1}\alpha^{n-1}$$

with each $a_i \in F$. But there are q choices for each a_i so a total of q^n elements in $F(\alpha)$.

31.

a. $x^3 + 2x + 1$ is irreducible

b. Thus $\mathbb{Z}_3[x]/(x^3+2x+1)$ has 27 elts by # 30.

36. F^* is a group with finitely many elements say n. So $\alpha^n = 1$ $\forall \alpha \in F^*$

so all $\alpha \in F$ are algebraic

37. Every element is algebraic by #36. Chose $\alpha \in F$, $\alpha \notin \mathbb{Z}_p$. Then $|F(\alpha)| = p^q$. If $F(\alpha) = F$ we are done. If not, choose $\alpha' \in F$, $\alpha' \notin F(\alpha)$. Then $|F(\alpha,\alpha')| = (p^q)^n$.

Keep doing this. F is finite so at some point $F = F(\alpha, \alpha', \dots, \alpha^{(\dots)})$ has p^n elts.