p. 207

6. $a_0 + a_1 x + a_2 x^2$ ~~w~~ with $a_0, a_1, a_2 \in \mathbb{Z}_5$

so $5^3 = \boxed{125}$ total

13. $x^3 + 2x + 2$ in $\mathbb{Z}_7$

0 no
1 no
2 yes $8+4+2 \equiv 0$
3 yes $27+6+2 \equiv 0$
4 ~~x~~ no
5 no
6 no

roots are 2, 3

check that $x^3+2x+2 = (x-2)^2(x-3)$

21. we need six polynomials w/ 5 a root

$(x-5)$, $(x-5)^2$, $x(x-5)$, $(x+1)(x-5)$, $(x-1)(x-5)$

22. $(2x^2+1)(2x^2+1) = 4x^4 + 4x + 1 \equiv 1$

so $2x^2+1$ is a unit

23. a. T b. T c. T d. T

e. F f. F g. T h. T i. T j. F

24. Let D be an integral domain. $D[x]$ is still commutative and 1 is still the identity so we must prove D has no zero divisors

Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ $g(x) = b_m x^m + \cdots + b_1 x + b_0 \in D[x]$

with $a_n \neq 0$, $b_m \neq 0$

Then $f(x)g(x) = a_n b_m x^{n+m}$ + lower degree terms

$a_n b_m \neq 0$ since D is an int. domain.

Thus $f(x)g(x) \neq 0$ so $D[x]$ is an int. domain.

25. a. units in $D[x]$ are units in D

b. $\{\pm 1\}$

c. $\{1, 2, 3, 4, 5, 6\}$

27. a. $D(p(x) + q(x)) = (p(x) + q(x))' = p'(x) + q'(x) = Dp + Dq$

so D is a group homo.

$D(fg) \neq Df\, Dg$ so not a ring hom.

b. Kernel = constant polys $\cong F$

c. Image is $F[x]$

p 218

9. x^4+4 roots: 1, 4, 2, 3 so in $\mathbb{Z}_5[x]$

$$x^4-4=(x-1)(x-4)(x-2)(x-3)$$

10. x^3+2x^2+2x+1 in $\mathbb{Z}_7[x]$

roots: 2,

$$\begin{array}{r} x^2+4x+3 \\ x-2\;\big)\;\overline{x^3+2x^2+2x+1} \\ \underline{x^3-2x^2} \\ 4x^2 \\ \underline{4x^2-8x} \\ 10x+1 \\ \underline{3x-\cancel{6}} \\ 0 \end{array}$$

x^2+4x+3 has 6 as a root

$$\begin{array}{r} x+3 \\ x-6\;\big)\;\overline{x^2+4x+3} \\ \underline{x^2-6x} \\ 10x+3 \\ \underline{3x-12} \\ 0 \end{array}$$

$$\boxed{x^3+2x^2+2x+1=(x-2)(x-6)(x+3)}$$

11. $2x^3+3x^2-7x-5$ in $\mathbb{Z}_{11}[x]$

3 is a root

$$\begin{array}{r} 2x^2+9x+9 \\ x-3\;\big)\;\overline{2x^3+3x^2-7x-5} \\ \underline{2x^3-6x^2} \\ 9x^2-7x \\ \underline{9x^2-27x} \\ 9x-5 \\ \underline{9x-27} \\ 0 \end{array}$$

$2x^2+9x+9$ has 8 as a root

$$\begin{array}{r} 2x+4 \\ x-8\;\big)\;\overline{2x^2+9x+9} \\ \underline{2x^2-16x} \\ 25x+9 \\ \underline{6x-29} \\ 0 \\ 4x+9 \\ \underline{4x-32} \end{array}$$

11. $2x^3+3x^2-7x-5$ in $\mathbb{Z}_{11}[x]$

3 is a root

$$\begin{array}{r} 2x^2+9x+9 \\ x-3 \overline{\big) 2x^3+3x^2-7x-5} \\ \underline{2x^3-6x^3} \\ 9x^2-7x \\ \underline{9x^2-27x} \\ 9x-5 \\ \underline{9x-27} \\ 0 \end{array}$$

$2x^2+9x+9$ has 8 as a root

$$\begin{array}{r} 2x+3 \\ x-8 \overline{\big) 2x^2+9x+9} \\ \underline{2x^2-16x} \\ 3x+9 \\ \underline{3x-24} \\ 0 \end{array}$$

$$\boxed{(x-3)(x-8)(2x+3)}$$

12. x^3+2x+3 in $\mathbb{Z}_5[x]$ is irred. iff no roots

2 is a root! Not irreducible

$$\begin{array}{r} x^2+2x+1 \leftarrow (x+1)^2 \\ x-2 \overline{\big) x^3+2x+3} \\ \underline{x^3-2x^3} \\ 2x^2+2x \\ \underline{2x^2-4x} \\ x+3 \\ \underline{x-2} \\ 0 \end{array}$$

$(x-2)(x+1)^2$

$\ldots$ in $\mathbb{Z}_3[x]$

NO ROOTS + DEG $\leq 3 \Rightarrow$ IRREDUCIBLE

18 x^2-12 irr. BY EC $p=3$

19 $8x^3+6x^2-9x+24$ irr by EC $p=3$

20. $4x^{10}-9x^3+24x-18$ NO p works since $9 \mid 18$

21 $2x^{10}-25x^3+10x^2-30$ irr by EC $p=5$

25. a. T b T c T d. F $x^2+3=(x+5)(x+2)$

e T f T g T h T i T j T

26 We want all p such that -2 is a root of

$x^4+x^3+x^2-x+1$ in $\mathbb{Z}_p[x]$

$(-2)^4+(-2)^3+(-2)^2-(-2)+1=16-8+4+2+1=15$

so $\boxed{p=3,5}$

27. $\cancel{0}$ $\quad$ $\cancel{1}$

$\cancel{0+x}$ $\quad$ $\cancel{1+x}$

$0+x^2$ $\quad$ $\cancel{1+x^2}$

$\cancel{x+x^2}$ $\quad$ $1+x+x^2$

8 total, 4 are degree 2, only $\boxed{1+x+x^2}$ is irred.

~~30.~~

p 243

4. $2\mathbb{Z}/8\mathbb{Z}$

cosets $0+8\mathbb{Z}$

$2+8\mathbb{Z}$

$4+8\mathbb{Z}$

$6+8\mathbb{Z}$

+	0	2	4	6
0	0	2	4	6
2	2	4	6	0
4	4	6	0	2
6	6	0	2	4

•	0	2	4	6
0	0	0	0	0
2	0	4	0	4
4	0	0	0	0
6	0	4	0	4

No! $\mathbb{Z}_4$ has no element which gives a row of zeros in mult-table, $2\mathbb{Z}/8\mathbb{Z}$ does

12. $\mathbb{Z}/7\mathbb{Z}$ ← I.D. — Field

13. $\mathbb{Z}/8\mathbb{Z}$ $\quad$ $\mathbb{Z}$ is I.D. $\quad$ $\mathbb{Z}/8\mathbb{Z}$ has zero divisors

15. $\{(q,a) \mid a \in \mathbb{Z}\}$

22. a. $\phi: R \to R'$ homo, N ideal in R.

a. Let $\phi(n) \in \phi[N]$, $\phi(r) \in \phi[R]$.

$\phi(n)\phi(r) = \phi(nr) \in \phi(N)$ since $nr \in N$
$\phi(r)\phi(n) = \phi(rn) \in \phi(N)$ since $rn \in N$.

Thus $\phi(N)$ is an ideal

c. Let $r \in \phi^{-1}[N']$ so $\phi(r) \in N'$.

Let $x \in R$.

Then $\phi(xr) = \phi(x)\phi(r) \in N'$ since N' is an ideal

Thus $xr \in \phi^{-1}[N']$.

Similar proof shows $rx \in \phi^{-1}[N']$

so $\phi^{-1}[N']$ is an ideal

26. $I_a = \{x \in R \mid ax = 0\}$

Let $x \in I_a$, $r \in R$. $a(xr) = axr = 0$ so $\underset{\underset{rx}{\parallel}}{xr} \in I_a$

Thus I_a is an ideal

27. Let I_1, I_2 be ideals. Let $n \in I_1 \cap I_2$, $r \in R$

rn, $nr \in I_1$ since I_1 is an ideal
rn, $nr \in I_2$ " I_2 is an ideal

Thus $rn, nr \in I_1 \cap I_2$ so $I_1 \cap I_2$ is an ideal

30. Let $a^n = 0$ and $r \in R$

$(ar)^n = a^n r^n = 0$ since R is comm

Thus $\underset{\underset{ra}{\parallel}}{ar}$ is nilpotent so {nilpotents} is an ideal