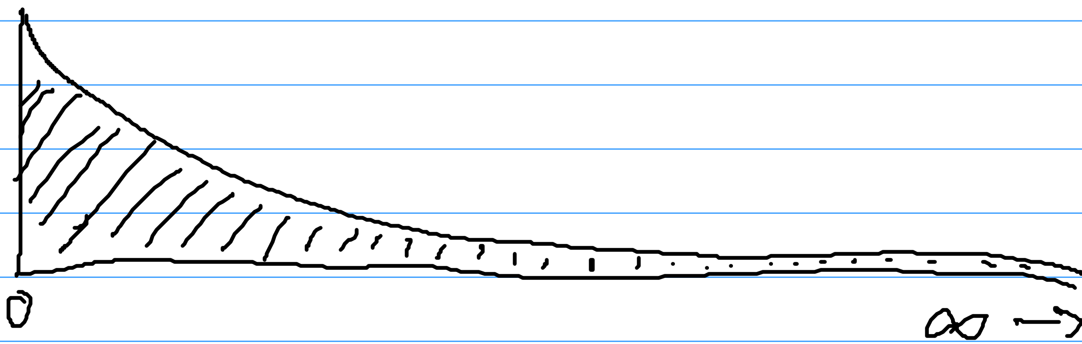
5.3 Improper Integrals (part 1)

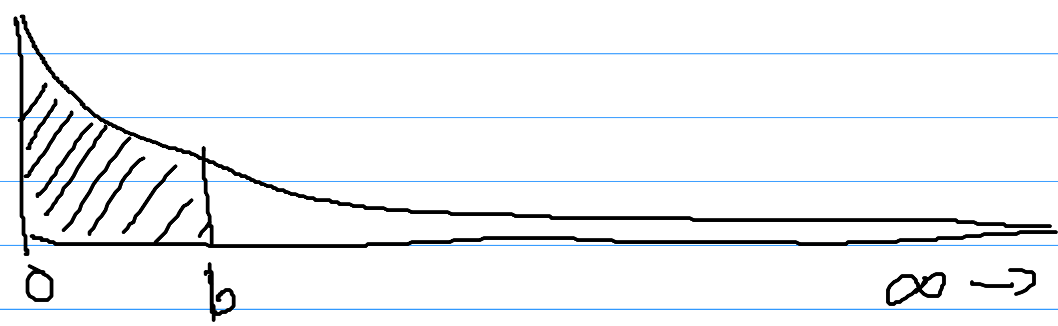
Consider $\int_0^\infty e^{-x}dx$

This is the area under e^{-x} from 0 to ∞.



Does the area over an infinite integral $[0,\infty)$ make sense?

Area over a finite interval does.



$$\text{area} = \int_0^b e^{-x}dx$$

As b gets larger, the area over $[0,b]$ gets closer to the "area" over $[0,\infty)$

If $f(x)$ is any function

$$\int_0^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_0^b f(x)\,dx$$

Ex $$\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$$

Limits involving $+\infty$ or $-\infty$ are called <u>improper intergrals</u>.

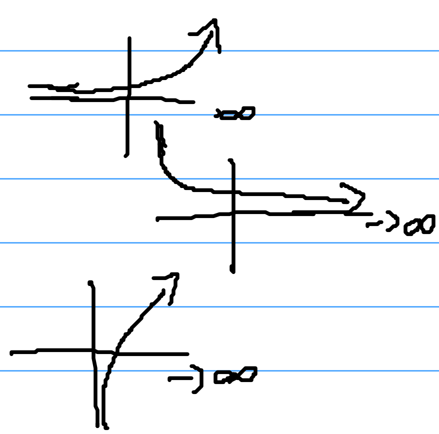
If the limit exists, then the improper integral is said to <u>converge</u>.

If the limit does not exist (or is $\pm\infty$), then the integral is said to <u>diverge</u>.

Common Limits that come up:

for $n>0$: $\lim_{b\to\infty} b^n$ diverges

$\lim_{b\to\infty} \frac{1}{b^n} = \lim_{b\to\infty} b^{-n} = 0$



$\lim_{b\to\infty} e^b$ diverges

$\lim_{b\to\infty} e^{-b} = 0$

$\lim_{b\to\infty} \ln(b)$ diverges

Common Limits Shortcut

- Negative exponent $\longrightarrow$ limit = 0
- otherwise diverges

Ex $\int_1^\infty 2x\,dx = \lim_{b\to\infty} \int_1^b 2x\,dx$

$= \lim_{b\to\infty} x^2 \Big|_1^b$

$= \lim_{b\to\infty} b^2 - 1^2$

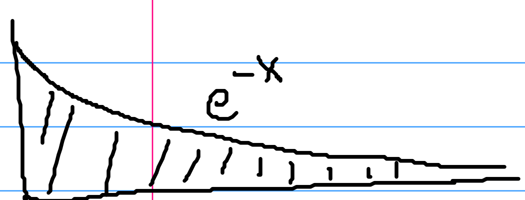
diverges

Ex $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$

$= \lim_{b\to\infty} -e^{-x} \Big|_0^b$

$= \lim_{b\to\infty} -e^{-b} - -e^{-0}$

converges



"area" = 1

$= 0 + e^0$

$= 1$

Ex $\int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x^2}\,dx$

$= \lim_{b\to\infty} \int_2^b x^{-2}\,dx$

$= \lim_{b\to\infty} -x^{-1}\Big|_2^b$

$= \lim_{b\to\infty} -b^{-1} - -2^{-1}$

converges

$= 0 + 2^{-1}$

$= \frac{1}{2}$

Ex $\int_2^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \ln x\Big|_2^b$

$= \lim_{b\to\infty} \ln b - \ln 2$

diverges