4.4 Properties of Definite Integrals (part 1)

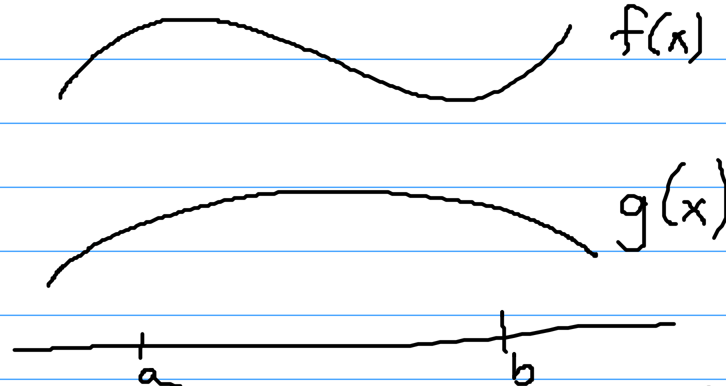
Recall: $\int f(x)$ = indef. integral = general antider.

$\int_a^b f(x)dx$ = def integral

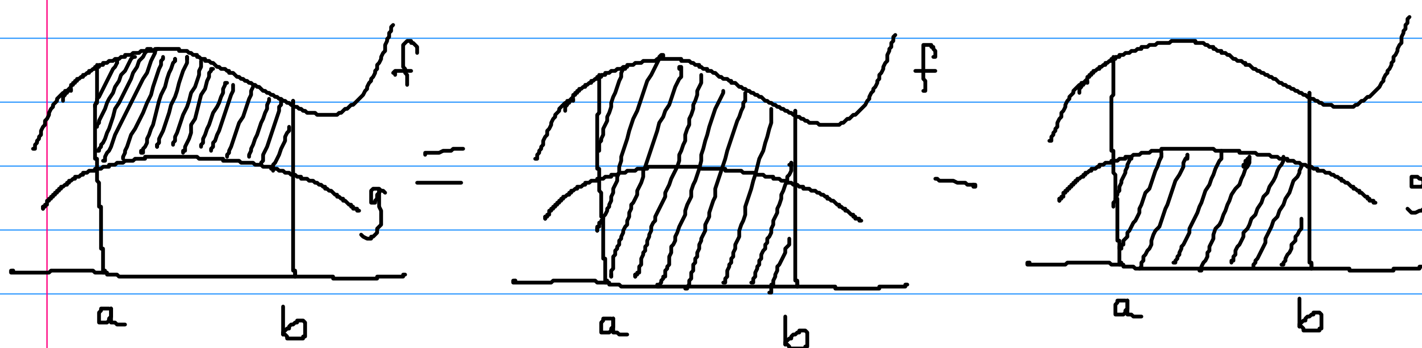
= area under f(x) from a to b

$= \text{anti} \Big|_a^b = F(b) - F(a)$

Suppose f(x) and g(x) are two curves over [a,b] with f(x) above g(x) on that interval.



The area between the curves can be found.



$$\text{area} = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$= \int_a^b [f(x) - g(x)]\, dx$$

$$\text{"="} \int_a^b [\text{top} - \text{bottom}]\, dx$$

Determine top/bottom via sketch or test points.

Ex Find the area between $f(x) = x^2 + 1$ and $g(x) = x-1$ from $x=0$ to $x=2$

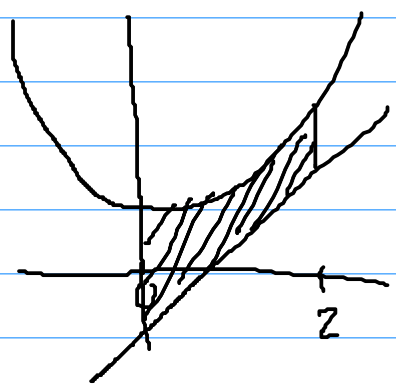
Top/bottom ?

Test on $x=1$ (between 0 & 2)

$f(1) = 1^2 + 1 = 2$ top

$g(1) = 1 - 1 = 0$ bottom

Sketch



$$\text{area} = \int_a^b \text{top} - \text{bottom}\ dx$$

$$= \int_0^2 (x^2+1) - (x-1)\ dx$$

$$= \int_0^2 x^2+1 - x + 1\ dx$$

$$= \int_0^2 x^2 - x + 2\ dx$$

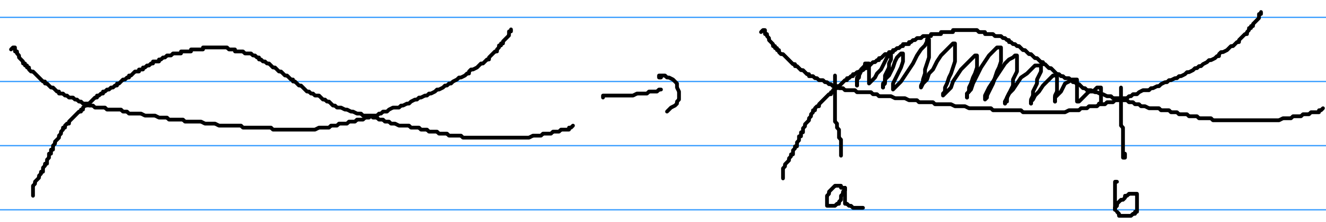
$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x \Big|_0^2$$

$$= \underbrace{\left[\tfrac{1}{3}(2)^3 - \tfrac{1}{2}(2)^2 + 2(2)\right]}_{F(2)} - \underbrace{\left[\tfrac{1}{3}(0)^3 - \tfrac{1}{2}(0)^2 + 2(0)\right]}_{F(0)}$$

$$= \left[8/3 - 2 + 4\right] - \left[0 - 0 + 0\right]$$

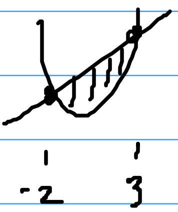
$$= 8/3 + 2 = 8/3 + 6/3 = 14/3$$

When endpoints are not given, find where the curves intersect and use those points as endpoints.



Ex Find the area in the closed region between

$$y = x^2 - 4 \quad \text{and} \quad y = x + 2$$



Find endpoints:

$$x^2 - 4 = x + 2$$
$$x^2 - x - 6 = 0$$
$$(x+2)(x-3) = 0$$
$$x = -2, 3 \quad \Leftarrow a, b$$

Top/bottom:

test point: $x = 0$ (between -2 & 3)

$x^2 - 4$: $(0)^2 - 4 = -4$ bottom

$x + 2$: $(0) + 2 = 2$ top

$$\text{area} = \int_a^b \text{top} - \text{bottom} \, dx$$
$$= \int_{-2}^{3} (x+2) - (x^2 - 4) \, dx$$
$$= \int_{-2}^{3} x + 2 - x^2 + 4 \, dx$$
$$= \int_{-2}^{3} -x^2 + x + 6 \, dx$$

$$= -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x \Big|_{-2}^{3}$$

$$= F(3) - F(-2)$$

$$= \left[-\frac{1}{3}(3)^3 + \frac{1}{2}(3)^2 + 6(3)\right] - \left[-\frac{1}{3}(-2)^3 + \frac{1}{2}(-2)^2 + 6(-2)\right]$$

$$= \left[-9 + \frac{9}{2} + 18\right] - \left[+\frac{8}{3} + 2 - 12\right]$$

$$= \left[\frac{9}{2} + 9\right] - \left[\frac{8}{3} - 10\right]$$

$$= \left[\frac{9}{2} + \frac{18}{2}\right] - \left[\frac{8}{3} - \frac{30}{3}\right]$$

$$= \frac{27}{2} - -\frac{22}{3}$$

$$= \frac{27}{2} + \frac{22}{3}$$

$$= \frac{81}{6} + \frac{44}{6}$$

$$= \frac{125}{6}$$