1.6 Differentiation Techniques:
The Product and Quotient Rules (part 2)

Power Rule $\frac{d}{dx} x^n = n x^{n-1}$

Product Rule $(u \cdot v)' = u'v + uv'$

Quotient Rule $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Product Rule Example:

$f(x) = (5x - 7)^2 = \underbrace{(5x-7)}^{u}\underbrace{(5x-7)}^{v}$

Find f'(x)

$u = 5x - 7 \qquad v = 5x - 7$

$u' = 5 \qquad v' = 5$

$$\begin{aligned} f'(x) &= u'v + uv' \\ &= 5(5x-7) + (5x-7)5 \\ &= 25x - 35 + 25x - 35 \\ &= 50x - 70 \end{aligned}$$

Quotient Rule Example:

$$f(x) = \frac{x^2 + 7x + 2}{x} \qquad \frac{u}{v}$$

Find f'(x)

$u = x^2 + 7x + 2 \qquad v = x$

$u' = 2x + 7 \qquad v' = 1$

$$f'(x) = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$= \frac{(2x+7)x - (x^2+7x+2)1}{x^2}$$

$$= \frac{2x^2+7x - (x^2+7x+2)}{x^2}$$

$$= \frac{2x^2+7x - x^2-7x-2}{x^2}$$

$$= \frac{x^2-2}{x^2}$$

Non-QR method (just in this case!) A check

$$f(x) = \frac{x^2}{x} + \frac{7x}{x} + \frac{2}{x}$$

$$= x + 7 + 2x^{-1}$$

$$f'(x) = 1 \quad -2x^{-2}$$

$$= 1 - \frac{2}{x^2}$$

$$= \frac{x^2}{x} - \frac{2}{x^2} = \frac{x^2-2}{x^2}$$

Quotient Rule Example:

$$f(x) = \frac{4}{x^2 + 3x - 8} \qquad \frac{u}{v}$$

Find f'(x)

$u = 4 \qquad v = x^2 + 3x - 8$

$u' = 0 \qquad v' = 2x + 3$

$$f'(x) = \frac{u'v - uv'}{v^2}$$

$$= \frac{0(x^2+3x-8) - 4(2x+3)}{(x^2+3x-8)^2}$$

$$= \frac{0 - (8x+12)}{(x^2+3x-8)^2}$$

$$= \frac{-8x-12}{(x^2+3x-8)^2}$$

Average Cost per Unit and the rate that it changes

If the cost to produce x units is $C(x)$
then the average cost per unit is $\frac{C(x)}{x}$

Denote it A_c

Similarly,
$C(x)$ = cost $A_c = C(x)/x$ = avg. cost per unit
$R(x)$ = revenue $A_R = R(x)/x$ = avg. revenue per unit
$P(x)$ = profit $A_p = P(x)/x$ = avg. profit per unit

Also recall that P(x) = $R(x) - C(x)$

Example: The cost of producing x chairs is
$C(x) = 100 + 10\sqrt{x}$
The revenue for selling x chairs is
$R(x) = 30\sqrt{x}$

Find the average cost per unit.

$$A_c = \frac{C(x)}{x} = \frac{100+10\sqrt{x}}{x}$$

Find the average profit per unit

$$P(x) = R(x) - C(x) = 30\sqrt{x} - (100 + 10\sqrt{x}) = 20\sqrt{x} - 100$$

$$A_p = \frac{P(x)}{x} = \frac{20\sqrt{x} - 100}{x}$$

means derivative

The rate that the average cost, revenue, and ~~price~~ profit is changing is A_C' A_R' A_P'

Find the rate for which the average profit is changing when 36 chairs have been produced and sold.

$$A_p = \frac{20\sqrt{x} - 100}{x} \qquad \frac{u}{v}$$

Find A_p' $\qquad u = 20\sqrt{x} - 100 \qquad v = x$

$u' = \frac{10}{\sqrt{x}} \qquad v' = 1$

subproblem $u = 20\sqrt{x} - 100$, Find u'

$u = 20x^{1/2} - 100$

$u' = 20 \cdot \frac{1}{2} x^{-\frac{1}{2}}$

$= 10x^{-\frac{1}{2}} = \frac{10}{x^{1/2}} = \frac{10}{\sqrt{x}}$

$\left(\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}\right)$

$$A_p' = \frac{u'v - uv'}{v^2}$$

$$= \frac{\frac{10}{\sqrt{x}} \cdot x - (20\sqrt{x} - 100) \cdot 1}{x^2}$$

$\frac{x}{\sqrt{x}} = \sqrt{x}$

$\frac{x^1}{x^{1/2}} = x^{1/2}$

$$= \frac{10\sqrt{x} - 20\sqrt{x} + 100}{x^2}$$

$$= \frac{-10\sqrt{x} + 100}{x^2}$$

For $x = 36$

$$= \frac{-10\sqrt{36} + 100}{36^2} = \frac{-10 \cdot 6 + 100}{1296} = \frac{40}{1296} = \$0.0308/\text{unit}$$

For $x = 100$

$$= \frac{-10\sqrt{100} + 100}{100^2} = \frac{-10 \cdot 10 + 100}{10{,}000} = \$0/\text{unit}$$