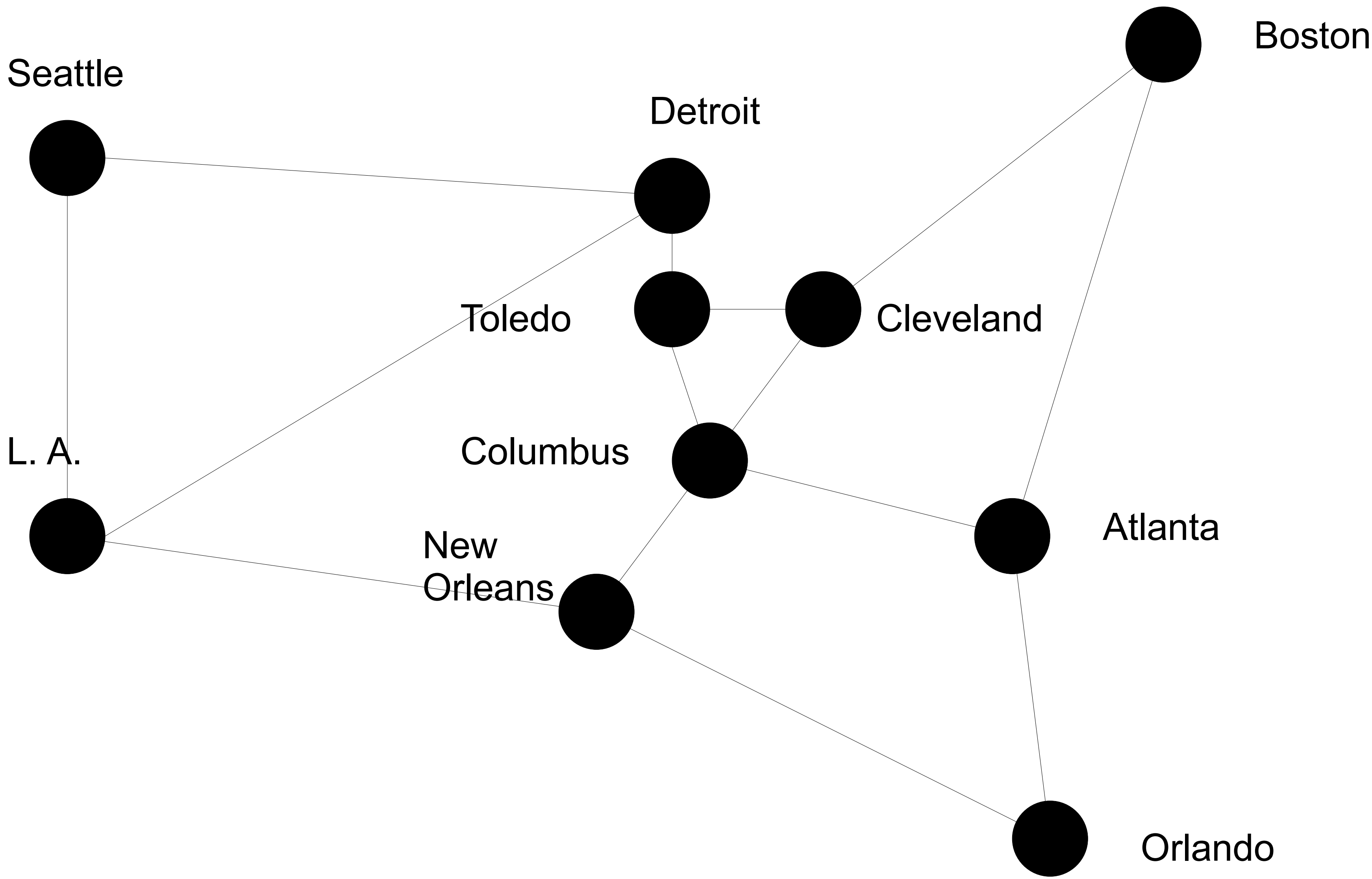
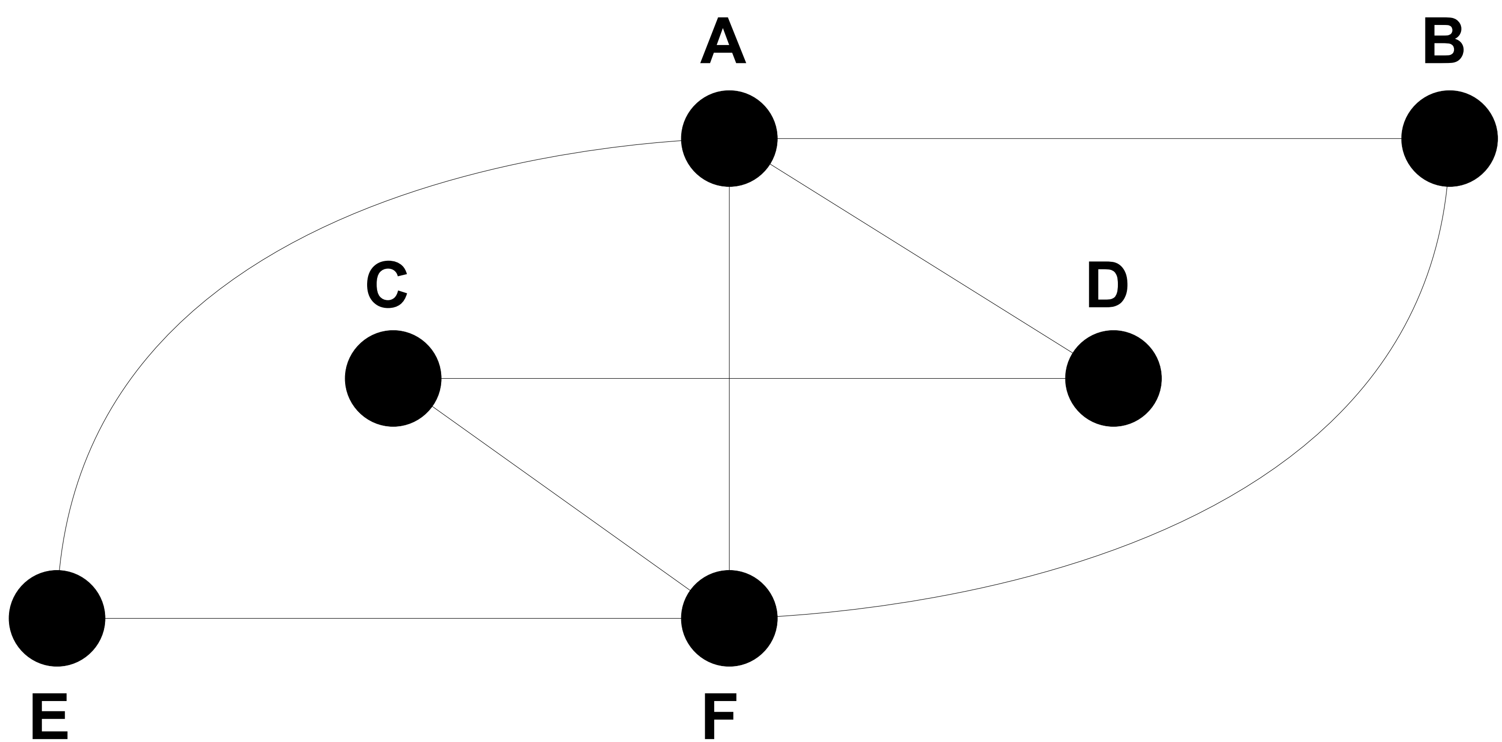
Traveling Salesman Problem (TSP)

- Visit every city and then go home.

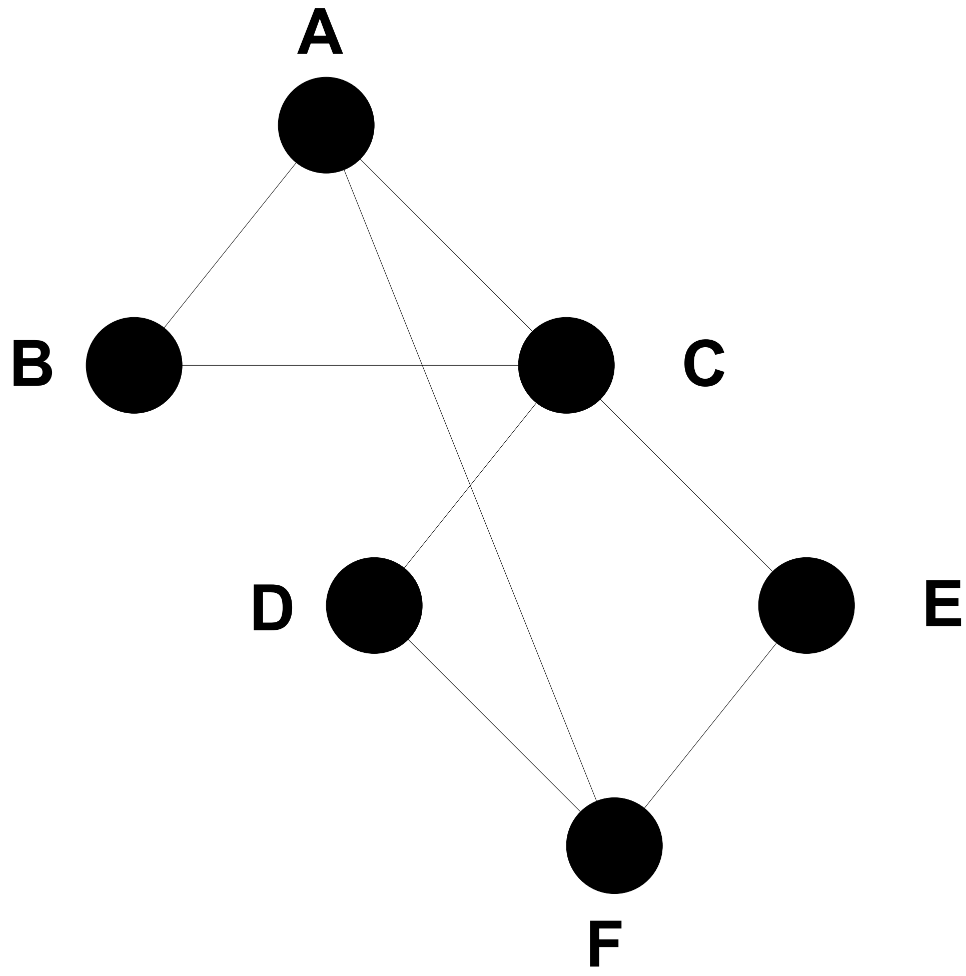

Seattle
Detroit
Boston
Toledo
Cleveland
L. A.
Columbus
Atlanta
New Orleans
Orlando

A **Hamiltonian Path** is a path that goes through each vertex exactly once.

Note: Not all edges have to be used.

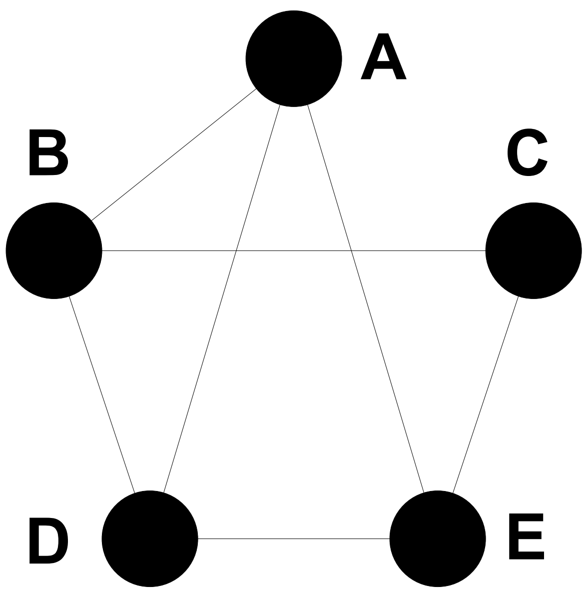


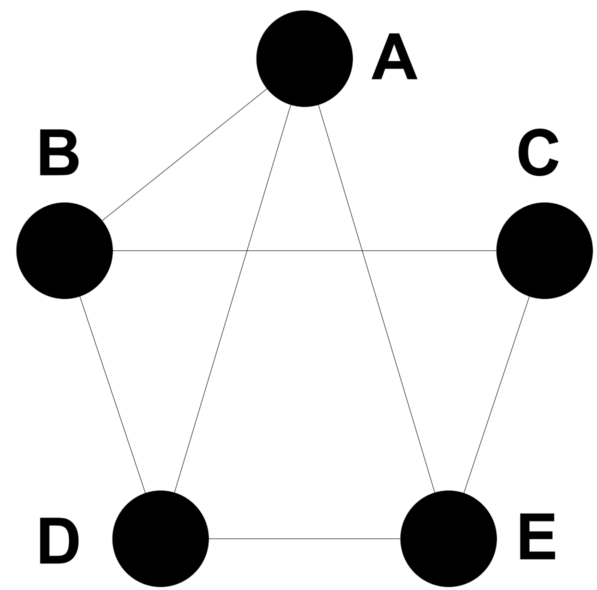
Does this graph have a Hamiltonian path?



A **Hamiltonian Circuit** is a Hamiltonian path that starts and ends at the same vertex.

This is the traveling salesman problem.





For any given graph, finding an Euler path will be easier than finding a Hamiltonian path.

Pop Quiz!

An Euler path visits every _____ once

1) vertex
2) edge
3) something else

PROBLEM SOLVING

The Splitting-Hairs Principle

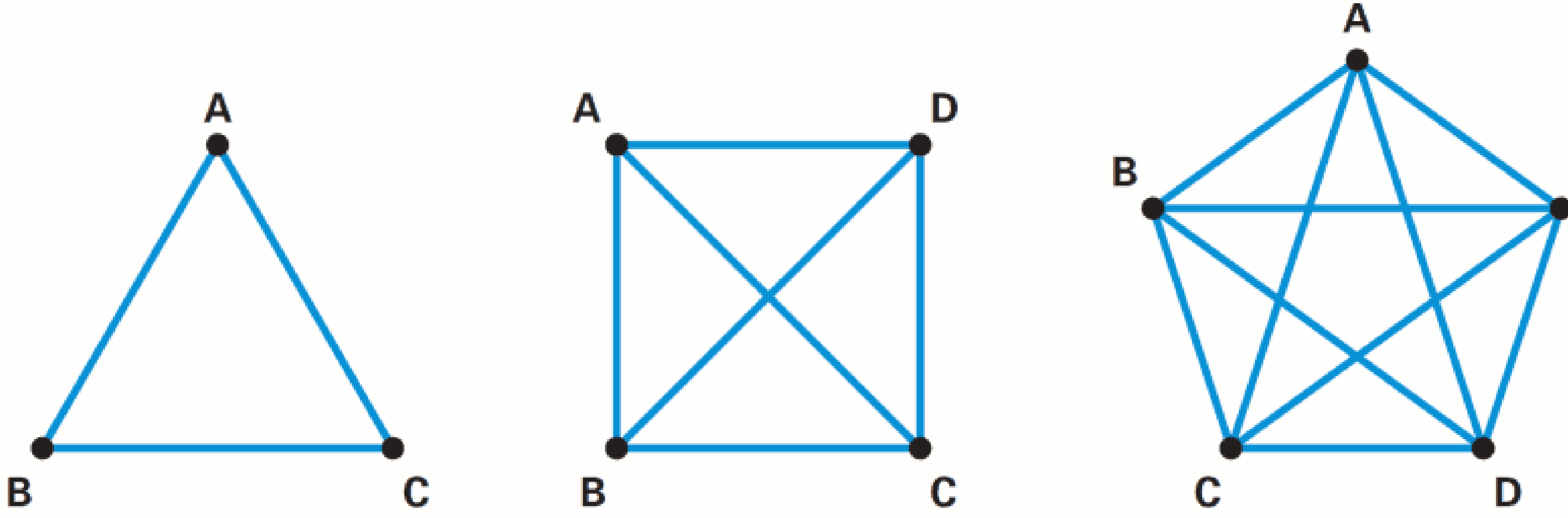
Recall the Splitting-Hairs Principle in Section 1.1. Although the definitions of *Hamilton path* and *Euler path* sound similar, they are not the same. In producing a Hamilton path, you do not have to trace every edge, as with an Euler path.

Euler path/circuit = cross each edge once

Hamiltonian path/circuit = cross each vertex once

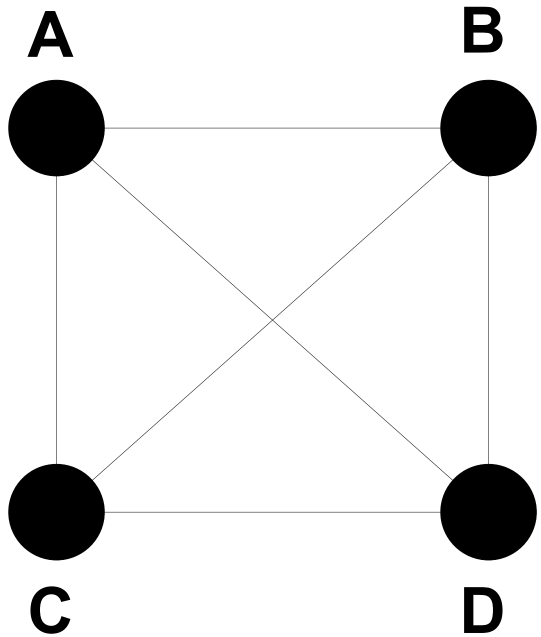
DEFINITION A **complete graph** is one in which every pair of vertices is joined by an edge. A complete graph with *n* vertices is denoted by K_n.*

- Examples:



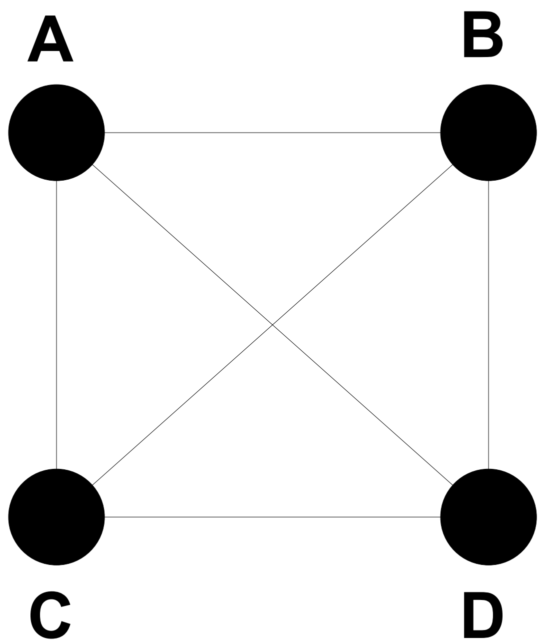
K_3 K_4 K_5

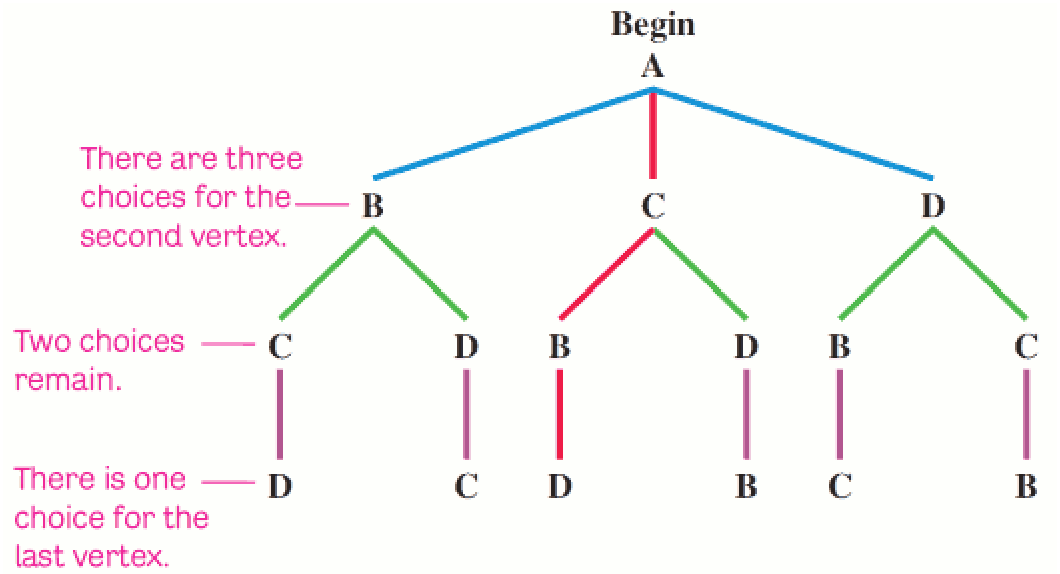
Let's find K_6

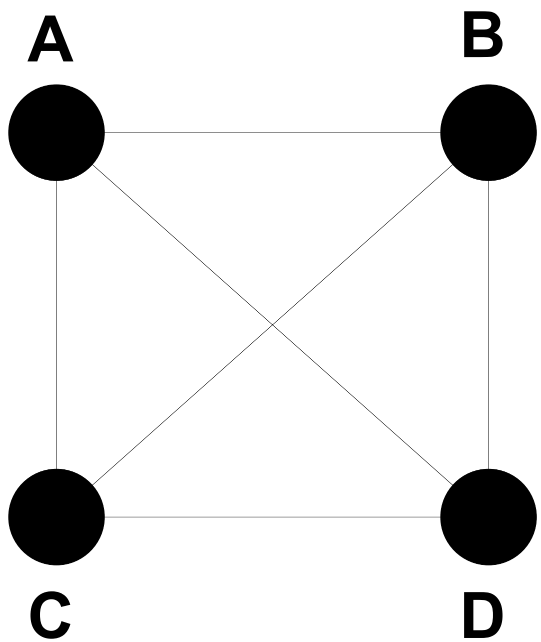


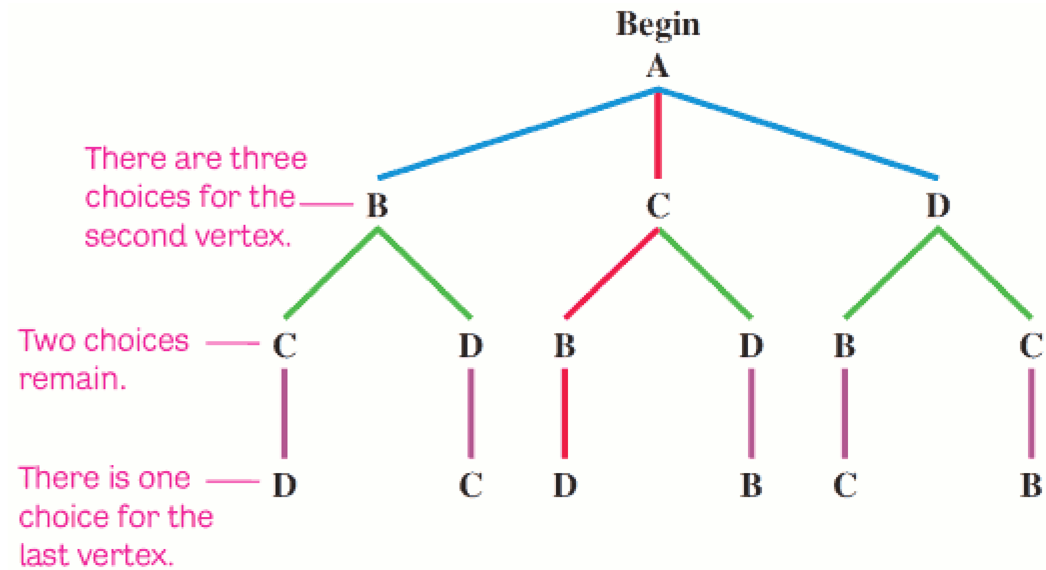
Since every vertex is visited, it doesn't matter where you start.

Let's start from A.


A
B
C
D


Begin
A
There are three choices for the second vertex.
B
C
D
Two choices remain.
C
D
B
D
B
C
There is one choice for the last vertex.
D
C
D
B
C
B





Tracing through the six branches of the tree, we see that K_4 has six Hamilton circuits:

reverse circuits

ABCDA, ABDCA, ACBDA, ACDBA, ADBCA, ADCBA

For K_5 we have 5 verticies A,B,C,D,E

Since we visit every vertex, we can start at A.

There are 4 verticies left. All can be reached.

From there 3 are left (all reachable).
Then 2 are left, then one.
Then we take the edge back to A.

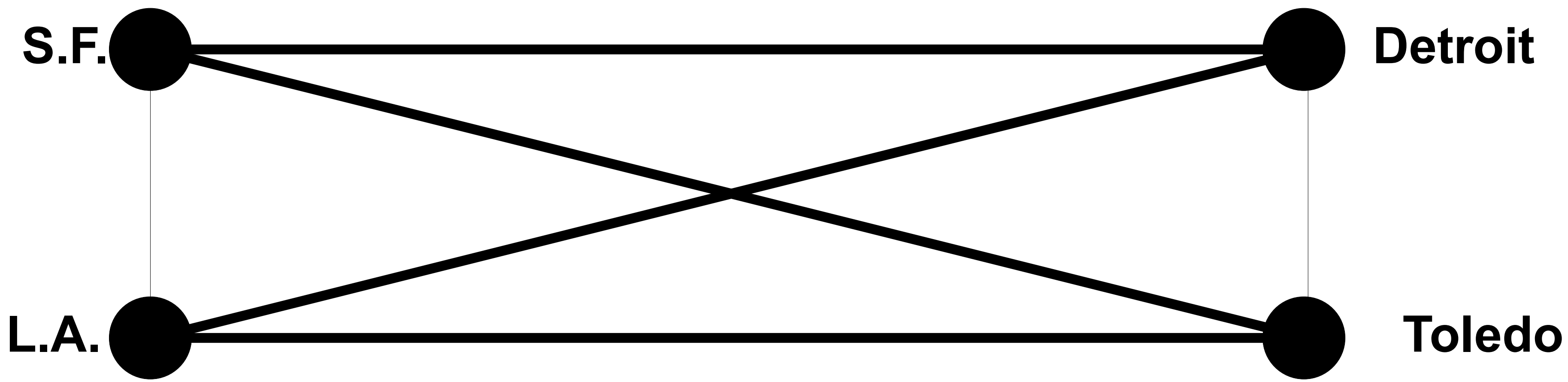
So there are 4x3x2x1=24 possible paths.

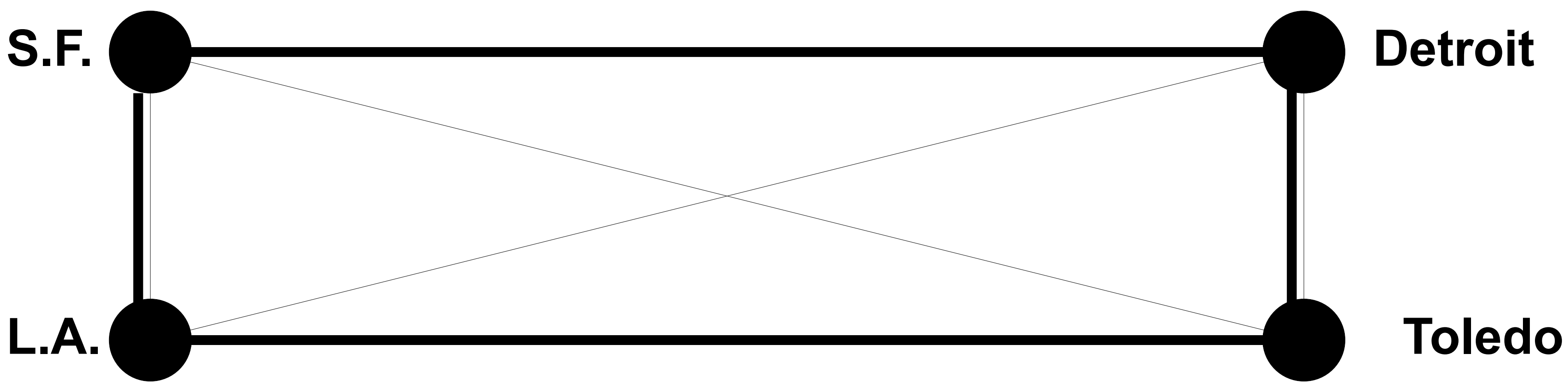
Finding Hamilton Circuits

THE NUMBER OF HAMILTON CIRCUITS IN K_n K_n has $(n-1)(n-2)(n-3)(n-4)\cdots 3\times 2\times 1$ Hamilton circuits. This number is written $(n-1)!$ and is called $(n-1)$ *factorial.*

n	Number of Hamilton Circuits in K_n
3	$2! = 2\cdot 1 = 2$
4	$3! = 3\cdot 2\cdot 1 = 6$
5	$4! = 4\cdot 3\cdot 2\cdot 1 = 24$
10	$9! = 362{,}880$
15	$14! = 87{,}178{,}291{,}200$
20	$19! = 121{,}645{,}100{,}408{,}832{,}000$

Which Hamiltonian circuit is better?



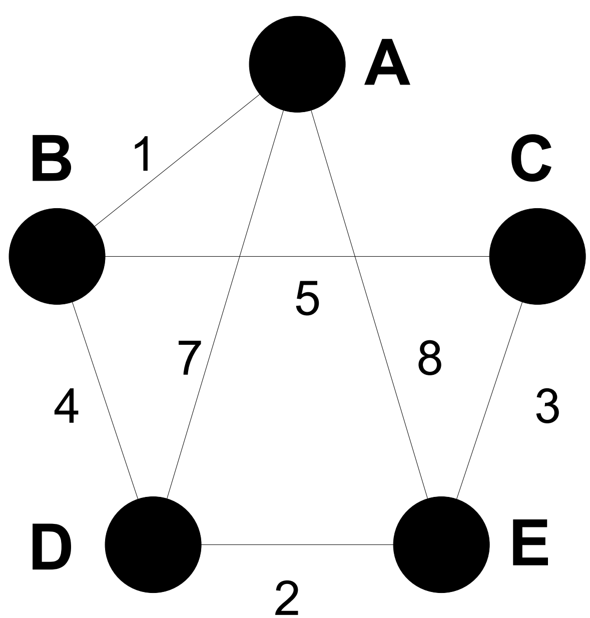


In a Traveling Salesman Problem, not only do you want to visit each city exactly once, but you also want to do so as *efficiently* as possible.

This will be measured by the total distance for the trip.

The **weight** of an edge is a number assigned to the edge. (Think distance between cities.)

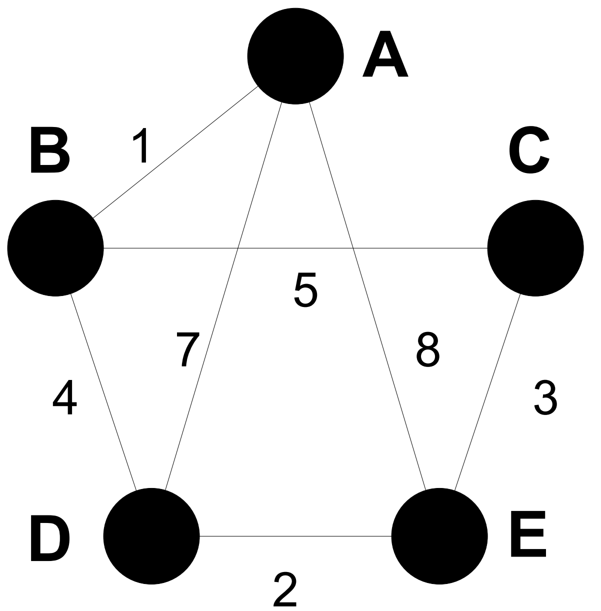
A graph is a **weighted graph** if all of its edges have weights.



What is the weight of

a) A,B
b) C,B
c) C,D - no edge, no weight

The weight of a path is the sum of the weights of the edges along the path.



What is the weight of the path B,A,D,E ?

Q: What is the relation between the weight of a path and the weight of its reverse path? Why?

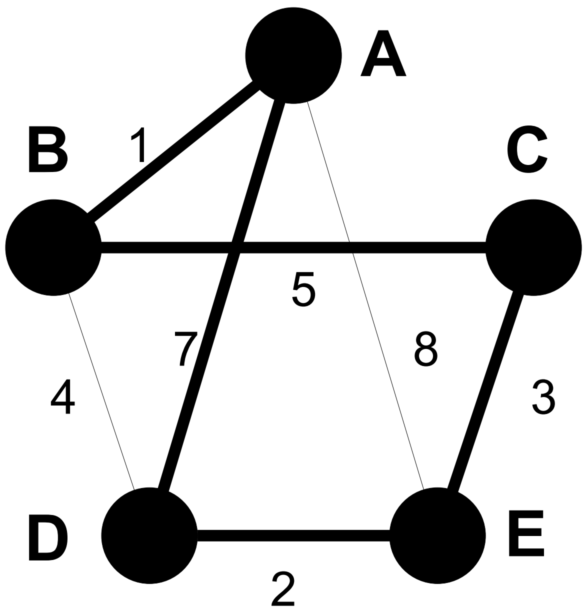
A:

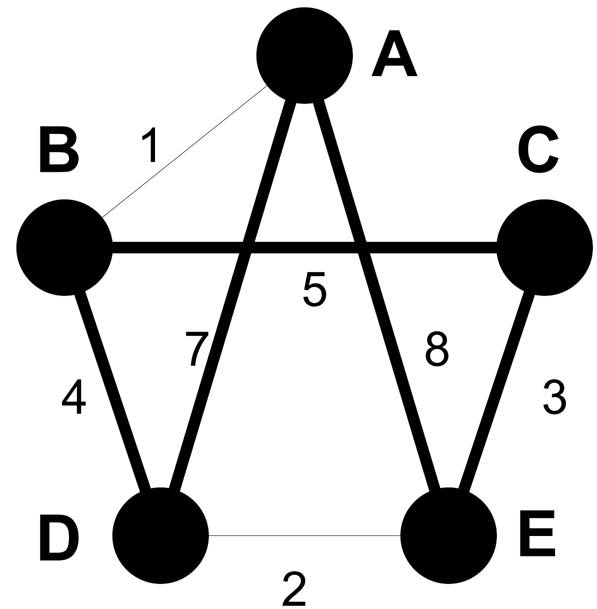
Q: What is the relation between the weight of a path and the weight of its reverse path?   Why?

A: The are the same.

The path and its reverse path use the same edges.

(So direction doesn't matter for finding the weight of a path.)





Which is the circuit with the smaller weight?

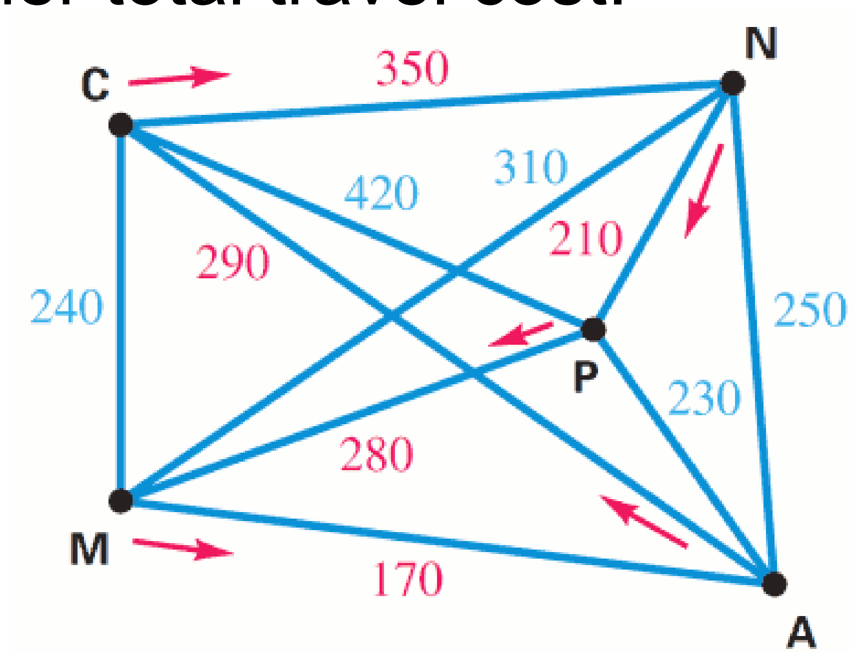
That one is the solution to the Traveling Salesman Problem.

Solving the TSP by Brute Force

- Brute Force means trying all possible outcomes and then picking the best one.
- In solving a TSP problem by brute force, we consider all possible Hamiltonian circuits.

Solving the TSP by Brute Force

- Example: Use the weighted graph to find the sequence of cities for Danielle to visit that will minimize her total travel cost.



280 (Philadelphia to Memphis) + 170 (Memphis to Atlanta) + 290 (Atlanta to Cleveland) + 350 (Cleveland to New York) + 210 (New York to Philadelphia) = $1,300 (Total)

Solving the TSP by Brute Force

- Solution: Use brute force to explore all possible circuits:

Hamilton Circuit	Weight ($)
PACMNP	1,280
PACNMP	1,460
PAMCNP	1,200
PAMNCP	1,480
PANCMP	1,350
PANMCP	1,450
PCAMNP	1,400
PCANMP	1,550
PCMANP	1,290
PCNAMP	1,470
PMACNP	1,300
PMCANP	1,270

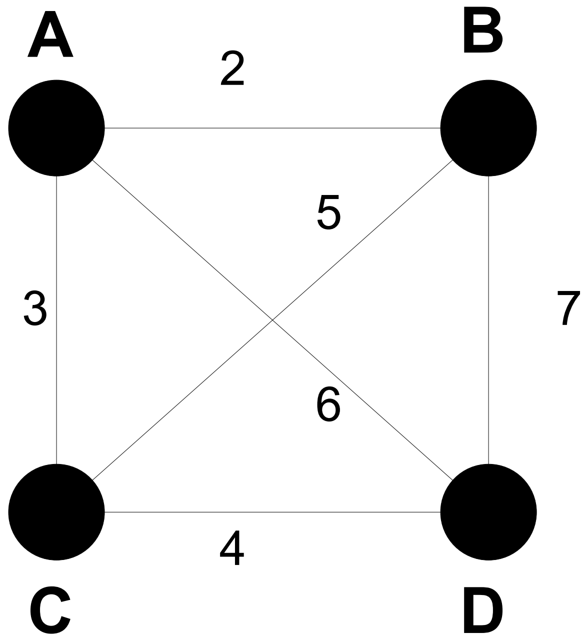
Solving the TSP by Brute Force

THE BRUTE FORCE ALGORITHM FOR SOLVING THE TSP

Step 1: List all Hamilton circuits in the graph.

Step 2: Find the weight of each circuit found in step 1.

Step 3: The circuits with the smallest weights tell us the solution to the TSP.



Paths are:

ABCDA (reverse ADCBA)

ACBDA (reverse ADBCA)

ABDCA (reverse ACDBA)

The Nearest Neighbor Algorithm

- There are algorithms that give good approximations to the TSP.

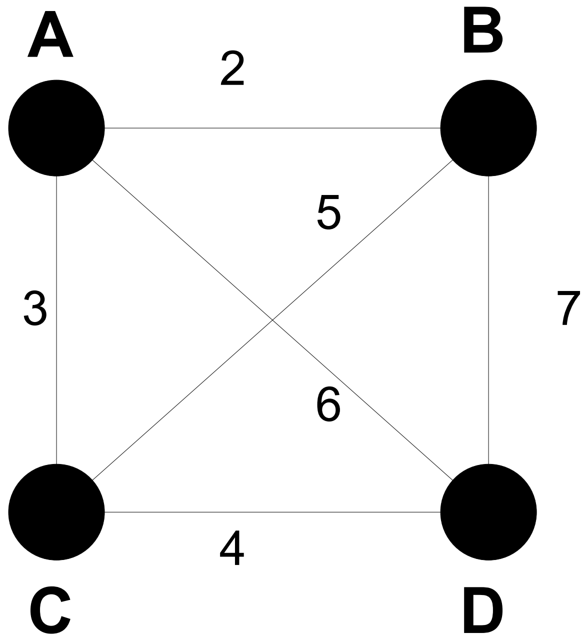
THE NEAREST NEIGHBOR ALGORITHM FOR SOLVING THE TSP

Step 1: Start at any vertex *X*.

Step 2: Of all the edges connected to *X*, choose any one that has the smallest weight. (There may be several with smallest weight.) Select the vertex at the other end of this edge. This vertex is called the *nearest neighbor* of *X*.

Step 3: Choose subsequent *new* vertices as you did in step 2. When choosing the next vertex in the circuit, choose one whose edge with the current vertex has the smallest weight.

Step 4: After all vertices have been chosen, close the circuit by returning to the starting vertex.



Start from A

B,C,D remain, B is closest.

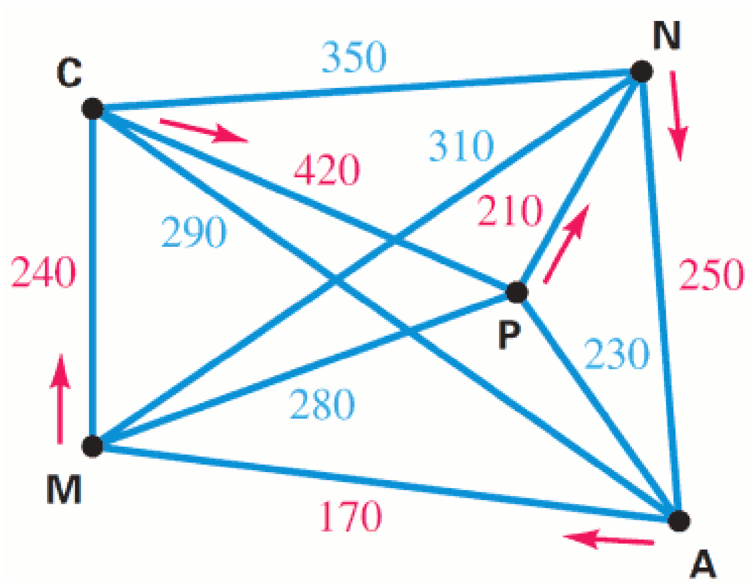
C,D remain, C is closest.

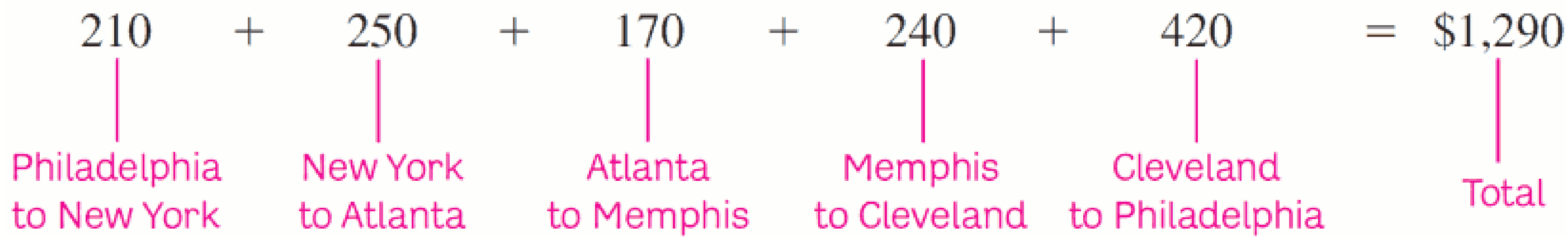
D remains.

Lastly go back to A.

The Nearest Neighbor Algorithm

- Example: Use the nearest neighbor algorithm to schedule Danielle's trip.
- Solution:





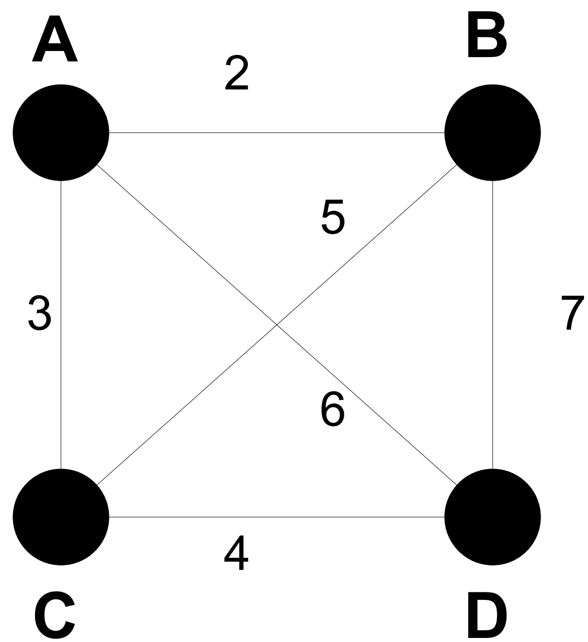
The Best Edge Algorithm

THE BEST EDGE ALGORITHM FOR SOLVING THE TSP

Step 1: Begin by choosing any edge with the smallest weight.

Step 2: Choose any remaining edge in the graph with the smallest weight.

Step 3: Keep repeating step 2; however, do not allow a circuit to form until all vertices have been used. Also, because the final Hamilton circuit cannot have three edges joined to the same vertex, never allow this to happen during the construction of the circuit.



2 is the smallest weight, add it.

3 is the next smallest, add it.

4 is the next smallest, add it.

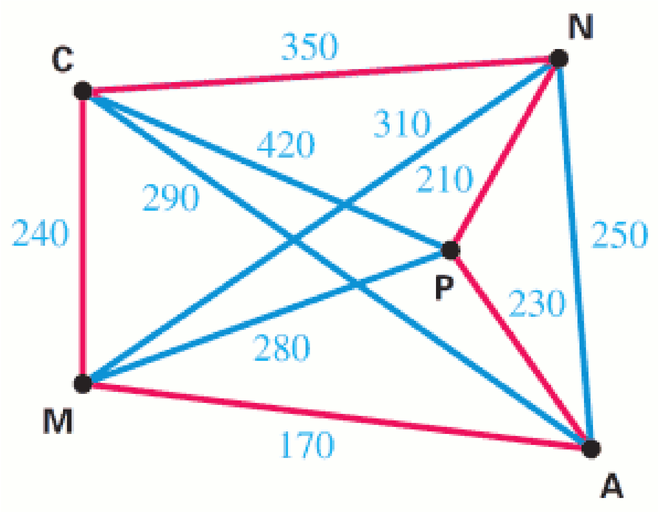
5 is next, can't add, triple at C.

6 is next, can't add, triple at A.

7 is next, add it and get the circuit!

The Best Edge Algorithm

- Example: Use the best edge algorithm to schedule Danielle's trip.
- Solution:



210 + 230 + 170 + 240 + 350 = 1200

Notice that this circuit has a weight of 1,200, which also makes it the best solution to Danielle's problem.