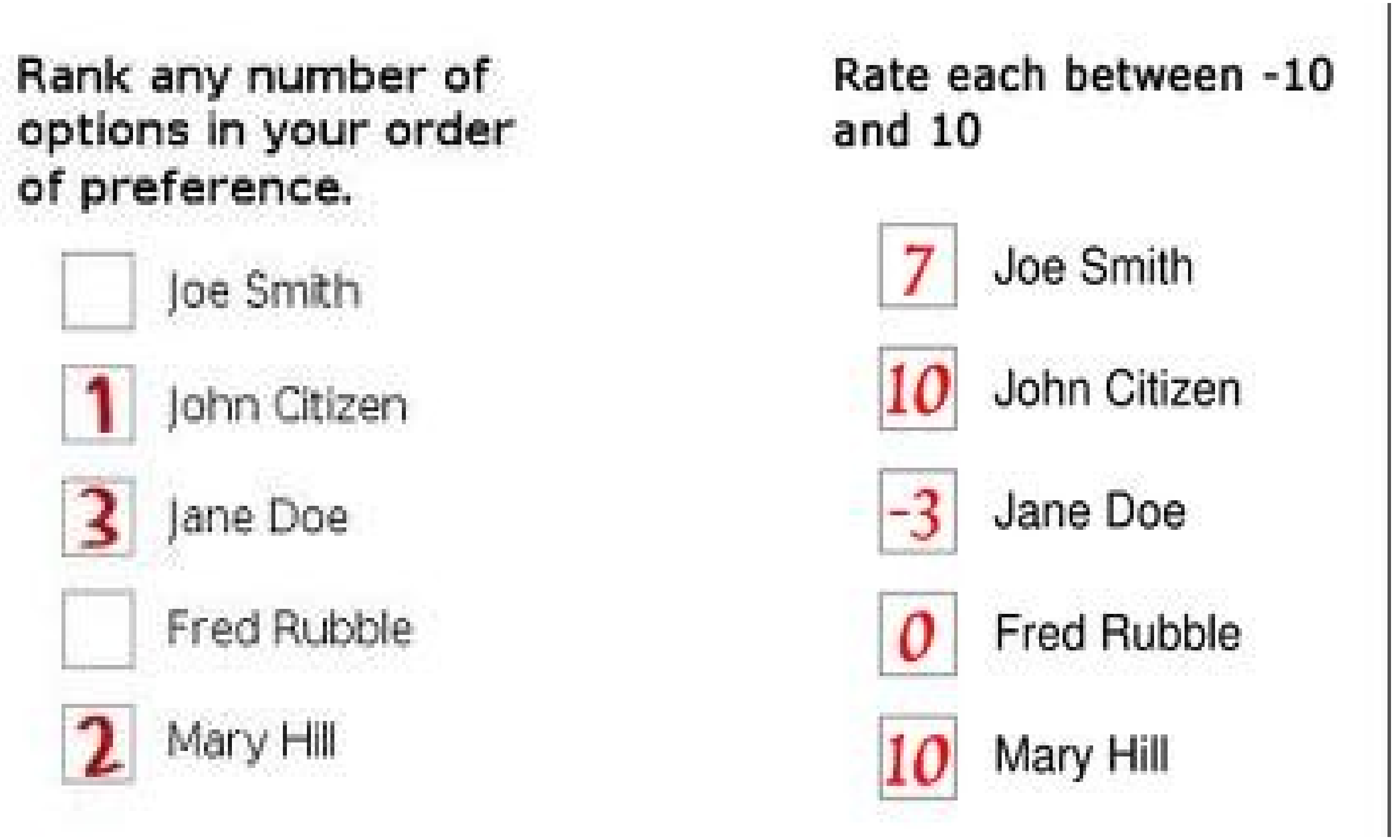
12.3 Weighted Voting Systems



There are different voting systems to the ones we've looked at.

Instead of focusing on the candidates, let's focus on the voters.

In a **weighted voting system**, the votes of some voters matters more than others.

Here, we will not have a “one person, one vote” principle.

Example: a stockholder with more shares has more of an effect on corporate policy than a stockholder with fewer shares.

The **weight** of a voter is the number of votes they have for an issue.

Examples:

- everyone has 1 vote.
- some have 5 votes, some have 2.
- stockholders have as many votes as they have shares.

A **quota** of votes is the number of votes need to get an issue passed.

Examples:

- Majority vote (more than ½)
- More than 2/3 of all votes.

- 15 votes out of a possible 30
- 20 votes out of a possible 30

DEFINITIONS A **weighted voting system** with n voters is described by a set of numbers that are listed in the following format:

$$[\text{quota: weight of voter 1, weight of voter 2, . . . , weight of voter } n]$$

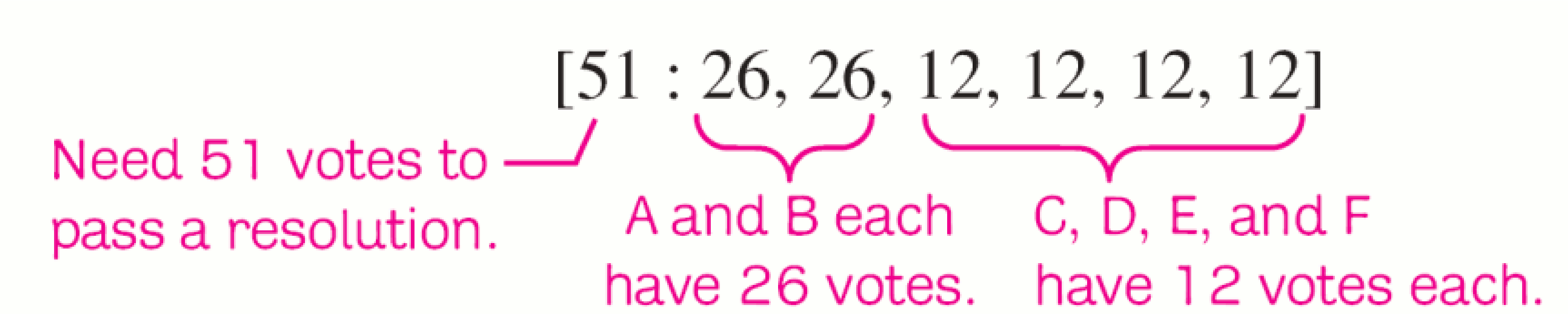
The **quota** is the number of votes necessary in this system to get a resolution passed. The numbers that follow, called **weights**, are the amount of votes controlled by voter 1, voter 2, etc.

Weighted Voting Systems

- Example: Explain the weighted voting system.

  [51 : 26, 26, 12, 12, 12, 12]

- Solution: The following diagram describes how to interpret this system.



Example:

[3 : 1, 1, 1, 1, 1]

Example:

[10: 2, 2, 2, 5, 5]

Example:

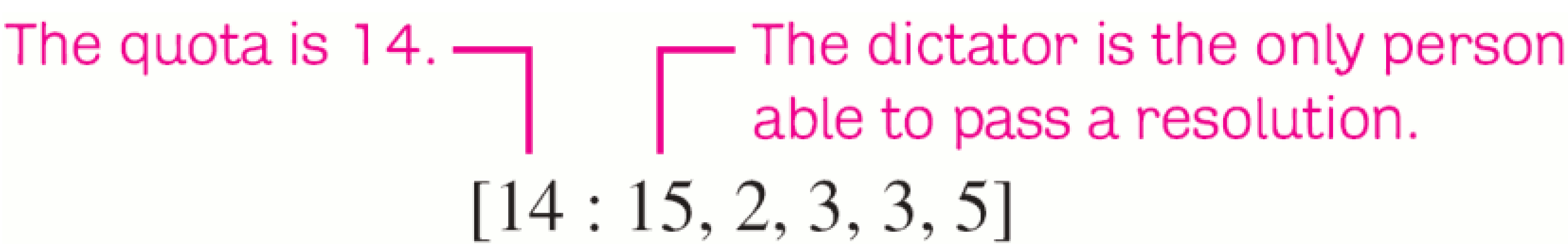
[100: 1, 2, 3, 2]

Weighted Voting Systems

- Example: Explain the weighted voting system.

  [14 : 15, 2, 3, 3, 5]

- Solution: Voter 1 is a *dictator*.



Example (Jury for a criminal case):

[12: 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]

Sometimes groups of voters always vote the same way.

Examples:

- sometimes with political parties.
- (many years ago) workers were told to vote a certain way or they would get fired.

Any voters that vote the same way is called a **coalition**.

The sum of the weights of a coalition is called the **weight of the coalition**.

If the weight of a coalition is the same or more than quota (the minimum needed votes), that coalition can always pass an issue. That coalition is called a **winning coalition**.

In a weighted voting system

$$[4 : 1, 1, 1, 1, 1, 1, 1]$$

any coalition of four or more voters is a winning coalition.

Example: Find a coalition of voters.

[10: 2, 2, 4, 4, 4]

Recall the subsets of a set.

The set {a,b,c} has subsets:
{}
{a} {b} {c}
{a,b} {a,c} {b,c}
{a,b,c}

Example: A quota of 8 votes.
There are 3 voting groups: A, B, C
A has 5 votes, B has 3 votes and C has 4 votes.
Which subsets of {A,B,C} are a winning coalition?

DEFINITION A voter in a winning coalition is called **critical** if it is the case that if he or she were to leave the coalition, then the coalition would no longer be winning.

Example: Who is critical to get 8 votes?

{A}	5	
{B}	3	
{C}	4	
{A,B}	8	winning
{A,C}	9	winning
{B,C}	7	
{A,B,C}	12	winning

Example: Quota of 10
Weights: R 9, D 8, I 3

Coalition	Weight		Critical Voters
{R}	9		
{D}	8		
{I}	3		
{R, D}	17	Winning	R, D
{R, I}	12	Winning	R, I
{D, I}	11	Winning	D, I
{R, D, I}	20	Winning	none

Remove any of these voters and the coalition no longer wins.

DEFINITION In a weighted voting system, a voter's **Banzhaf power index** is defined as

$$\frac{\text{the number of times the voter is critical in winning coalitions}}{\text{the total number of times voters are critical in winning coalitions}}.$$

Compute the Banzhaf Power Index for A, B, C.

			critical
{A}	5		
{B}	3		
{C}	4		
{A,B}	8	winning	A, B
{A,C}	9	winning	A, C
{B,C}	7		
{A,B,C}	12	winning	A

The Banzhaf Power Index

In the previous example, we saw that R, D, and I each were critical voters twice. Thus, R's Banzhaf power index is

$$\frac{\text{the number of times R is critical in winning coalitions}}{\text{the total number of times voters are critical in winning coalitions}} = \frac{2}{6} = \frac{1}{3}.$$

The Banzhaf Power Index

- Example: A law has two senior partners (Krooks and Cheatum) and four associates (W, X, Y, and Z). To change any major policy of the firm, Krooks, Cheatum, and at least two associates must vote for the change. Calculate the Banzhaf power index for each member of this firm.

- Need K and C, need at least 2 of W, X, Y, and Z

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The Banzhaf Power Index

- Solution: We use {K, C, W, X, Y, Z} to represent the firm. Since every winning coalition includes {K, C} and any two of the other associates, we only need to determine the subsets of {W, X, Y, Z} with two or more members to determine the winning coalitions.

2-Element Subsets of {W, X, Y, Z}	3-Element Subsets of {W, X, Y, Z}	4-Element Subsets of {W, X, Y, Z}
{W, X}, {W, Y}, {W, Z} {X, Y}, {X, Z}, {Y, Z}	{W, X, Y}, {W, X, Z}, {W, Y, Z}, {X, Y, Z}	{W, X, Y, Z}

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The Banzhaf Power Index

The winning coalitions and critical members are:

All voters are necessary to pass a resolution in these coalitions. (rows 1–6)

Only K and C are critical in these coalitions. (rows 7–11)

	Winning Coalitions	Critical Members
1	{K, C, W, X}	K, C, W, X
2	{K, C, W, Y}	K, C, W, Y
3	{K, C, W, Z}	K, C, W, Z
4	{K, C, X, Y}	K, C, X, Y
5	{K, C, X, Z}	K, C, X, Z
6	{K, C, Y, Z}	K, C, Y, Z
7	{K, C, W, X, Y}	K, C
8	{K, C, W, X, Z}	K, C
9	{K, C, W, Y, Z}	K, C
10	{K, C, X, Y, Z}	K, C
11	{K, C, W, X, Y, Z}	K, C

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The Banzhaf Power Index

K and C are critical members 11 times, whereas W, X, Y, and Z are each critical members only 3 times. We may compute the Banzhaf power index for each member.

Members	Banzhaf Power Index
K, C	$\frac{11}{11 + 11 + 3 + 3 + 3 + 3} = \frac{11}{34}$
W, X, Y, Z	$\frac{3}{11 + 11 + 3 + 3 + 3 + 3} = \frac{3}{34}$

The Banzhaf Power Index

- Example: A 5-person air safety review board consists of a federal administrator (A), two senior pilots (S and T), and two flight attendants (F and G). The intent is for the A to have considerably less power than S, T, F, and G, so A only votes in the case of a tie; otherwise, cases are decided by a simple majority. How much less power does A have than the other members of the board?

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The Banzhaf Power Index

- Solution: We see that each board member (including A) is a critical member of exactly six coalitions. All members have equal power.

	Winning Coalitions	Critical Members
1	{S, T, F}	S, T, F
2	{S, T, G}	S, T, G
3	{S, F, G}	S, F, G
4	{T, F, G}	T, F, G
5	{S, T, F, G}	None
6	{A, S, T}	A, S, T
7	{A, S, F}	A, S, F
8	{A, S, G}	A, S, G
9	{A, T, F}	A, T, F
10	{A, T, G}	A, T, G
11	{A, F, G}	A, F, G

No tie (rows 1–5)

Tie broken by administrator (rows 6–11)