Bayes Rule

Conditional probabilities are not in general commutative, that is

$$P(A|B) \neq P(B|A).$$

On occasion, we are given one of these probabilities, when what we really would like to know is the other.

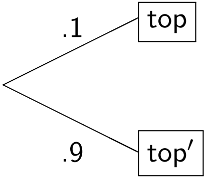
For example, suppose a parent is considering enrolling their 8th grader in the Great Readers reading improvement program. In the company's brochure, they are told that out of all the local 10th graders who had scored in the top 10% on a recent reading skills test, 35% had been through the Great Readers program. This might sound impressive, but it is incomplete information. Suppose you found out that 30% of the students who did *not* score in the top 10% on this reading exam had been through the Great Readers program. Suddenly 35% of the top 10% doesn't sound quite so good, does it?

The 35% figure is a conditional probability, i.e. the probability that a student went through the Great Readers program given that they scored in the top 10% on the reading skills test. We will denote this $P(GR|\mathrm{top})$. What we really want to know is the probability that a child who has been through the Great Readers program will score in the top 10%, i.e. $P(\mathrm{top}|GR)$.

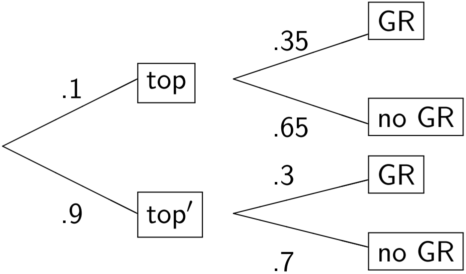
This type of problem (called a Bayes Rule prtoblem) can be approached using a tree diagram and the conditional probability rule

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We begin our tree diagram with the non-conditional probabilities



A student either scored in the top 10% or the didn't, so the probabilities must sum to 1. For each branch, we draw two more branches corresponding to whether or not they had been through the Great Readers program. We put the conditional probabilities here.



$P(\text{top} \cap \text{GR}) = .35 \times .1 = .035$

$P(\text{top} \cap \text{no GR}) = .65 \times .1 = .065$

$P(\text{top}' \cap \text{GR}) = .3 \times .9 = .27$

$P(\text{top}' \cap \text{no GR}) = .7 \times .9 = .63$

From this tree diagram and the Conditional Probability Rule, we can calculate $P(top|GR)$. Recall the conditional probability rule

$$P(\text{top}|\text{GR}) = \frac{\text{P(top } \cap \text{ GR)}}{\text{P(GR)}}$$

We've already calculated the numerator as part of our tree diagram. To get $P(GR)$, we add the probabilities for the first and third branches (the ones involving GR).

$$P(GR) = P(\text{top } \cap \text{ GR }) + P(\text{top}' \cap \text{ GR }) = .035 + .27 = .305$$

Then by the conditional probability rule

$$P(\text{top}|\text{GR}) = \frac{\text{P(top } \cap \text{ GR)}}{\text{P(GR)}} = \frac{.035}{.305} \approx .115$$

If the Great Readers program did not exist, the probability that a randomly chosen student would score in the top 10% would be, of course 10%, so 11.5% is not much of an improvement.