The Laplace Transform
Lecture 9

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9.1. Outline of Lecture

- Definition of the Laplace Transform.
- Solution of Initial Value Problems.

9.2. Definition of the Laplace Transform.

9.2.1. Improper Integrals.

We look into a brief overview of improper integrals. An improper integral over an unbounded interval is defined as a limit of integrals over finite intervals; thus

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt, \tag{9.1}$$

where A is a positive real number. If the integral from a to A exists for each $A > a$, and if the limit as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise the integral is said to **diverge**, or fail to exist.

We look at couple of examples.

Example 1. Let $f(t) = e^{ct}, t \geq 0$, where c is a real nonzero constant. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{A\to\infty}\int_0^A e^{ct}\,dt = \lim_{A\to\infty}\left.\frac{e^{ct}}{c}\right|_0^A \tag{9.2}$$

$$= \lim_{A\to\infty}\frac{1}{c}(e^{cA}-1). \tag{9.3}$$

It follows that the improper integral converges to the value $-1/c$ if $c < 0$ and diverges if $c > 0$. If $c = 0$, then $f(t)$ is the constant function with value 1, and the integral again diverges.

Example 2. Let $f(t) = 1/t, t \geq 1$. Then

$$\int_1^\infty \frac{dt}{t} = \lim_{A\to\infty} \int_1^A \frac{dt}{t} = \lim_{A\to\infty} \ln A. \tag{9.4}$$

Since $\lim_{A\to\infty} \ln A = \infty$, the improper integral diverges.

9.2.2. The Laplace Transform.

Among the tools that are very useful for solving linear differential equations are **integral transforms**. An integral transform is a relation of the form

$$F(s) = \int_\alpha^\beta K(s,t)f(t)\,dt, \tag{9.5}$$

where $K(s,t)$ is a given function, called the **kernel** of the transformation, and the limits of integration α and β are also given. The relation (9.5) transforms the function f into another function F, which is called the transform of f.

The Laplace transform of f, which we will denote by $\mathcal{L}\{f(t)\}$ or by $F(s)$, is defined by the equation

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)\,dt, \tag{9.6}$$

whenever the integral converges. The general idea in using the Laplace transform to solve a differential equation is as follows:

1. Use the relation (9.6) to transform an initial value problem for an unknown function f in the t-domain into a simpler problem (indeed, an algebraic problem) for F in the s-domain.
2. Solve this algebraic problem to find F.
3. Recover the desired function f from its transform F.

We look into an example where we find the Laplace transform.

Example 3. Find the Laplace transform of the function $f(t) = t$.

Solution 1.

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}t\,dt = \lim_{A\to\infty} \int_0^A e^{-st}t\,dt \tag{9.7}$$

We use integration by parts,

$$(9.8)\quad = \lim_{A\to\infty}\left[\frac{te^{-st}}{-s}\Bigg|_0^A + \frac{1}{s}\int_0^A e^{-st}\,dt\right] = \lim_{A\to\infty}\frac{Ae^{-sA}}{-s} - \frac{1}{s^2}\lim_{A\to\infty} e^{-st}\Bigg|_0^A$$

$$(9.9)\quad = -\frac{1}{s^2}\lim_{A\to\infty}\left[e^{-sA}-1\right] = \frac{1}{s^2},\quad s>0.$$

Therefore

$$(9.10)\quad \mathcal{L}\{t\} = \frac{1}{s^2},\quad s>0.$$

An important result before we wrap up this section. The Laplace transform is a **linear operator**, that is, if f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively. Then, for s greater than the maximum of a_1 and a_2,

$$(9.11)\quad \mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}.$$

9.3. Solution of Initial Value Problems.

In this section we show how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients. We present a theorem which we will use extensively in this section

Theorem 9.12. *Suppose that the functions* $f, f', \dots, f^{(n-1)}$ *are continuous and that* $f^{(n)}$ *is piecewise continuous on any interval* $0 \le t \le A$. *Suppose further that there exist constants* K, a *and* M *such that* $|f(t)| \le Ke^{at}, |f'(t)| \le Ke^{at}, \dots, |f^{(n-1)}(t)| \le Ke^{at}$ *for* $t \ge M$. *Then* $\mathcal{L}\{f^{(n)}(t)\}$ *exists for* $s > a$ *and is given by*

$$(9.13)\quad \mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

We look into an example where we solve an initial value problem using the method of Laplace transform.

Example 4. Find the solution of the differential equatio

$$(9.14)\quad y'' + y = \cos 2t.$$

satisfying the initial conditions

$$(9.15)\quad y(0) = 1,\qquad y'(0) = 0.$$

Solution 2. Taking the Laplace transform of the differential equation by using Theorem (9.12), we have

$$(9.16)\quad s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \mathcal{L}\{\cos 2t\}.$$

Using the table from the text book we have $\mathcal{L}\{\cos 2t\} = s/(s^2+4)$. Therefore

$$s^2\mathcal{L}\{y\} - s + \mathcal{L}\{y\} = \frac{s}{s^2+4} \tag{9.17}$$

$$\mathcal{L}\{y\}(s^2+1) = \frac{s}{s^2+4} + s$$

$$\mathcal{L}\{y\} = \frac{s}{(s^2+4)(s^2+1)} + \frac{s}{(s^2+1)} \tag{9.18}$$

Just like Laplace transform, the inverse Laplace transform is also a linear operator. Therefore

$$y = \mathcal{L}^{-1}\{\frac{s}{(s^2+4)(s^2+1)}\} + \mathcal{L}^{-1}\{\frac{s}{(s^2+1)}\} \tag{9.19}$$

Using partial fractions, we can write $s/(s^2+4)(s^2+1)$ in the form

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1} \tag{9.20}$$

Multiplying both sides of the above equation by $(s^2+4)(s^2+1)$ we have

$$s = (As+B)(s^2+1) + (Cs+D)(s^2+4). \tag{9.21}$$

Expanding the right side we have

$$s = s^3(A+C) + s^2(B+D) + s(A+4C) + (B+4D). \tag{9.22}$$

Equation coefficients of like powers of s, we have

$$A+C=0, \quad B+D=0, \quad A+4C=1, \quad B+4D=0. \tag{9.23}$$

Consequently, $A = -\frac{1}{3}, B = 0, C = \frac{1}{3}, D = 0$, from which it follows that

$$y = \mathcal{L}^{-1}\{-\frac{1}{3}(\frac{s}{s^2+4}) + \frac{1}{3}(\frac{s}{s^2+1})\} + \mathcal{L}\{\frac{s}{s^2+1}\}. \tag{9.24}$$

$$= \mathcal{L}^{-1}\{-\frac{1}{3}(\frac{s}{s^2+4})\} + \mathcal{L}^{-1}\{\frac{1}{3}(\frac{s}{s^2+1})\} + \mathcal{L}\{\frac{s}{s^2+1}\}. \tag{9.25}$$

Using the table from the text book we have

$$y = -\frac{1}{3}\cos 2t + \frac{1}{3}\cos t + \cos t = -\frac{1}{3}\cos 2t + \frac{4}{3}\cos t. \tag{9.26}$$

Therefore the solution of the given initial value problem is

$$y = -\frac{1}{3}\cos 2t + \frac{4}{3}\cos t. \tag{9.27}$$