Higher Order Linear Equations
Lecture 8

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8.1. Outline of Lecture

- The Method of Undetermined Coefficients.
- The Method of Variation of Parameters.

8.2. The Method of Undetermined Coefficients.

A particular solution Y of the nonhomogeneous nth order linear equation with constant coefficients

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t) \tag{8.1}$$

can be obtained by the method of undetermined coefficients, provided that $g(t)$ is of an appropriate form. This mimics the method of undetermined coefficients for second order nonhomogeneous equations (See Lecture 6). Thus if $g(t)$ is a polynomial $A_0 t^m + A_1 t^{m-1} + \cdots + A_m$, an exponential function $e^{\alpha t}$, a sine function $\sin \beta t$, cosine function $\cos \beta t$, or a combination of them, then our assumed solution is also a suitable combination of polynomials, exponentials, and so forth, multiplied by a number of undetermined constants. The constants are then determined by substituting the assumed expression into Eq. (8.1).

The main difference in using this method for higher order equations stems from the fact that roots of the characteristic polynomial equation may have multiplicity greater than 2. Consequently, terms proposed for the nonhomgeneous part of the solution may need to be multiplied by higher powers of t to make them different from terms in the solution of the corresponding homogeneous equation. We look at this case in the example below.

Example 1. Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t. \tag{8.2}$$

Solution 1. The characteristic polynomial for the homogeneous equation corresponding is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3, \tag{8.3}$$

Since the root 1 is repeated three times, therefore the general solution of the homogeneous equation is

$$y_c(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t. \tag{8.4}$$

To find a particular solution $Y(t)$, we start by assuming that $Y(t) = Ae^t$. However since e^t, te^t, and t^2e^t are all solutions of the homogeneous equation, we must multiply this initial choice by t^3. Thus our final assumption is that $Y(t) = At^3e^t$, where A is an undetermined coefficient.

We differentiate $Y(t)$ three times, substitute for y and its derivative in Eq. (8.2), and collect terms in the resulting equation. In this way we obtain

$$6Ae^t = 4e^t. \tag{8.5}$$

Therefore $A = \frac{2}{3}$ and the particular solution is

$$Y(t) = \frac{2}{3} t^3 e^t. \tag{8.6}$$

Therefore the general solution of Eq. (8.2) is given by

$$y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t. \tag{8.7}$$

8.3. The Method of Variation of Parameters.

The method of variation of parameters for determining a particular solution of the nonhomogeneous nth order linear differential equation

$$L[y] = y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_{n-1}(t) y' + p_n(t) y = g(t) \tag{8.8}$$

is a direct extension of the method for the second order differential equation that was covered in Lecture 6.

We look into the general theorem which illustrates the method.

Theorem 8.9. *If the functions $p_1, \ldots, p_n$ and g are continuous on an open interval I, and if the functions $y_1, y_2, \ldots, y_n$ are a fundamental set of solutions of the homogeneous equation corresponding to the nonhomogeneous equation (8.8), then a particular solution of Eq. (8.8) is*

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s) W_m(s)}{W(s)}\, ds, \tag{8.10}$$

where $W(t) = W(y_1, y_2, \ldots, y_n)(t)$, t_0 is any conveniently chosen point in I and W_m is the determinant obtained from W by replacing the mth column by the column $(0, 0, \ldots, 0, 1)$. The general solution is

$$y = c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t) + Y(t). \tag{8.11}$$

Let's look into an example which uses the above theorem.

Example 2. Use the method of variation of parameters to determine the general solution of the given differential equation.

$$y''' - y' = t \tag{8.12}$$

Solution 2. The characteristic polynomial of the corresponding homogeneous equation of Eq. (8.12) is

$$r^3 - r = 0 \tag{8.13}$$

The roots of this equation are $0, 1$ and -1. Therefore the solutions of the homogeneous equation are $1, e^t$ and e^{-t}.

$$W(1, e^t, e^{-t}) = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 2 \tag{8.14}$$

Hence $y_1(t) = 1, y_2(t) = e^t$ and $y_3(t) = e^{-t}$ form a fundamental set of solution. Therefore

$$W_1(t) = \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 1 & e^t & e^{-t} \end{vmatrix} = -2 \tag{8.15}$$

$$W_2(t) = \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & 1 & e^{-t} \end{vmatrix} = e^{-t} \tag{8.16}$$

$$W_3(t) = \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & 1 \end{vmatrix} = e^t \tag{8.17}$$

We choose $t_0 = 0$. Using the above theorem

$$Y(t) = 1 \cdot \int_0^t \frac{s(-2)}{2}\, ds + e^t \int_0^t \frac{se^{-s}}{2}\, ds + e^{-t} \int_0^t \frac{se^s}{2}\, ds \tag{8.18}$$

We evaluate the last two integrals using integration by parts. This gives

$$Y(t) = -\frac{t^2}{2} - 1 + \frac{e^{-t}}{2}. \tag{8.19}$$

Since 1 and e^{-t} are already solutions of the homogeneous equation, therefore the particular solution of Eq. (8.12) is $Y(t) = -\frac{t^2}{2}$.

Therefore the general solution of Eq. (8.12) is

$$y = c_1 + c_2 e^t + c_3 e^{-t} - \frac{t^2}{2}. \tag{8.20}$$