Higher Order Linear Equations
Lecture 7

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7.1. Outline of Lecture

- General Theory of nth Order Linear Equations.
- Homogeneous Equations with Constant Coefficients.

7.2. General Theory of nth Order Linear Equations

An nth order linear differential equation is an equation of the form

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t). \tag{7.1}$$

Since the equation involves the $n$the derivative of y, therefore to obtain a unique solution, it is necessary to specify n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)}. \tag{7.2}$$

The mathematical theory associated with Eq. (7.1) is completely analogous to that for the second order linear equation. Therefore we simply state the results for the nth order problem.

Theorem 7.3. *If the functions $p_1, p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation (7.1) that also satisfies the initial conditions (7.2). The solution exists throughout the interval I.*

7.2.1. The Homogeneous Equation.

As in the corresponding second order problem, we first discuss the homogeneous equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0. \tag{7.4}$$

If the functions $y_1, y_2, \ldots, y_n$ are solutions of Eq. (7.4), then it follows by direct computation that the linear combination

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t), \tag{7.5}$$

where $c_1, \ldots, c_n$ are arbitrary constants, is also a solution of Eq. (7.4).

We define the Wronskian of the solutions $y_1, \ldots, y_n$ by the determinant

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{7.6}$$

Theorem 7.7. *If the functions $p_1, p_2, \ldots, p_n$ are continuous on the open interval I, if the functions $y_1, y_2, \ldots, y_n$ are solutions of Eq. (7.4), and if $W(y_1, y_2, \ldots, y_n)(t) \neq 0$ for at least one point in I, then every solution of Eq. (7.4) can be expresses as a linear combination of the solutions $y_1, y_2, \ldots, y_n$.*

A set of solutions $y_1, \ldots, y_n$ of Eq. (7.4) whose Wronskian is nonzero is referred to as a **fundamental set of solutions**. Since all solutions of Eq. (7.4) are of the form (7.5), we use the term **general solution** to refer to any arbitrary linear combination of any fundamental set of solutions of Eq. (7.4).

7.2.2. Linear Dependence and Independence.

We now explore the relationship between fundamental sets of solutions and the concept of linear independence.

The functions $f_1, f_2, \ldots, f_n$ are said to be **linearly dependent** on an interval I if there exists a set of constants $k_1, k_2, \ldots, k_n$, not all zero, such that

$$k_1 f_1(t) + k_2 f_2(t) + \cdots + k_n f_n(t) = 0 \tag{7.8}$$

for all t in I. The functions $f_1, \ldots, f_n$ are said to be **linearly independent** on I if they are not linearly dependent there. We look into an example.

Example 1. Determine whether the functions $f_1(t) = 1, f_2(t) = 2 + t, f_3(t) = 3 - t^2$, and $f_4(t) = 4t + t^2$ are linearly independent or dependent on any interval I.

Solution 1. We form the linear combination

$$\begin{aligned} k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) + k_4 f_4(t) &= k_1 + k_2(2+t) + k_3(3-t^2) + k_4(4t+t^2) \\ &= (k_1 + 2k_2 + 3k_3) + (k_2 + 4k_4)t + (-k_3 + k_4)t^2. \end{aligned}$$

This expression is zero throughout an interval provided that

$$k_1 + 2k_2 + 3k_3 = 0, \quad k_2 + 4k_4 = 0, \quad -k_3 + k_4 = 0.$$

These three equations, with four unknowns, have many nontrivial solutions. For instance, if $k_4 = 1$, then $k_3 = 1, k_2 = -4$, and $k_1 = 5$. Thus the given functions are linearly dependent on every interval.

We now present the theorem describing the relation between linear independence and fundamental sets of solutions.

Theorem 7.9. *If $y_1(t), \ldots, y_n(t)$ is a fundamental set of solutions of Eq. (7.4)*

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0 \tag{7.10}$$

on an interval I, then $y_1(t), \ldots, y_n(t)$ are linearly independent on I. Conversely, if $y_1(t), \ldots, y_n(t)$ are linearly independent solutions of Eq. (7.4) on I, then they form a fundamental set of solutions of I.

7.2.3. The Nonhomogeneous Equation.

Consider the nonhomogeneous equation (7.1)

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t). \tag{7.11}$$

It follows that any solution of the above equation can be written as

$$y = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) + Y(t), \tag{7.12}$$

where $y_1, \ldots, y_n$ is fundamental set of solutions of the corresponding homogeneous equation and Y is some particular solution of the nonhomogeneous equation (7.1). The linear combination (7.12) is called the general solution of the nonhomogeneous equation (7.1).

7.3. Homogeneous Equations with Constant Coefficients

Consider the nth order linear homogeneous differential equation

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0, \tag{7.13}$$

where $a_0, a_1, \ldots, a_n$ are real constants. From our knowledge of second order linear equations with constant coefficients, it is natural to anticipate that $y = e^{rt}$ is a solution of Eq. (7.13) for suitable values of r. Indeed,

$$L[e^{rt}] = e^{rt}(a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n) = e^{rt}Z(r) \tag{7.14}$$

for all r, where

$$Z(r) = a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n. \tag{7.15}$$

The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation (7.13). A polynomial of degree n has n zeros, say $r_1, r_2, \ldots, r_n$, some of which may be equal; hence we can write the characteristic polynomial in the form

$$Z(r) = a_0(r - r_1)(r - r_2) \cdots (r - r_n). \tag{7.16}$$

Now we look at all the three possibilities of the nature of the roots.

7.3.1. Real and Unequal Roots.

If the roots of the characteristic equation are real and no two are equal, then we have n distinct solutions $e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}$ of Eq. (7.13). If these functions are linearly independent(check Wronskian), then the general solution of Eq. (7.13) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}. \tag{7.17}$$

7.3.2. Complex Roots.

If the characteristic equation has complex roots, they must occur in conjugate pairs, $\lambda \pm i\mu$, since the coefficients $a_0, \ldots, a_n$ are real numbers. Provided that none of the roots are repeated, the general solution of Eq. (7.13) is still of the form (7.17). Similar to the second order equation, we can replace the complex valued solutions $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$ by the real-valued solutions

$$e^{\lambda t} \cos \mu t, \quad e^{\lambda t} \sin \mu t \tag{7.18}$$

7.3.3. Repeated Roots.

If the roots of the characteristic equation are not distinct, that is if some of the roots are repeated, then we have to look at the multiplicity of the root. For an equation of order n, if a root of $Z(r) = 0$, say $r = r_1$, has multiplicity s (where $s \leq n$), then

$$e^{r_1 t}, \quad t e^{r_1 t}, \quad t^2 e^{r_1 t}, \quad \ldots, \quad t^{s-1} e^{r_1 t} \tag{7.19}$$

are corresponding solutions of Eq. (7.13).

If a complex root $\lambda + i\mu$ is repeated s times, the complex conjugates $\lambda - i\mu$ is also repeated s times. Corresponding to these $2s$ complex valued solutions, we can find $2s$ real valued solutions by noting that the real and imaginary parts of $e^{(\lambda+i\mu)t}, te^{(\lambda+i\mu)t}, \ldots, t^{s-1}e^{(\lambda+i\mu)t}$ are also linearly independent solutions:

$$\begin{gathered} e^{\lambda t} \cos \mu t, \quad e^{\lambda t} \sin \mu t, \quad t e^{\lambda t} \cos \mu t, \quad t e^{\lambda t} \sin \mu t, \\ \ldots, \quad t^{s-1} e^{\lambda t} \cos \mu t, \quad t^{s-1} e^{\lambda t} \sin \mu t. \end{gathered}$$

Let's look into an example below.

Example 2. Find the general solution of the given differential equation.

$$y''' - 3y'' + 7y' - 5y = 0. \tag{7.20}$$

Solution 2. The characteristic equation of the above differential equation is given by

$$Z(r) = r^3 - 3r^2 + 7r - 5 = 0 \tag{7.21}$$

Substituting $r = 1$, it can be verified that $Z(1) = 0$, hence $r = 1$ is a root of $Z(r)$. Since $(r-1)$ is a factor of $Z(r)$, hence by the Factor Theorem, the other factor can be found by dividing $Z(r)$ by $(r-1)$. The other factor is $r^2 - 2r + 5$ whose roots are $1 \pm 2i$. Hence the three roots of Eq. (7.20) are

$$e^t, \quad e^t \cos 2t, \quad e^t \sin 2t. \tag{7.22}$$

Therefore the general solution of Eq. (7.20) is given by

$$y = c_1 e^t + c_2 e^t \cos 2t + c_3 e^t \sin 2t. \tag{7.23}$$

for arbitrary constants c_1, c_2, c_3.