

Second Order Differential Equations

Lecture 5

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5.1. Outline of Lecture

- Solution of Linear Homogeneous Equations; the Wronskian
- Complex roots of the Characteristic Equation

5.2. Solution of Linear Homogeneous Equations; the Wronskian

In this lecture we provide a clearer picture of the structure of the solutions of all second order linear homogeneous equations using results from previous lectures. We will be asking some basic questions about second order linear homogeneous equations and answer them with the help of some theorems. Before doing that, let's define the notion of a *differential operator*.

Let p and q be continuous functions on an open interval I . Then for any twice differentiable function ϕ on I , we define the **differential operator** L by the equation

$$(5.1) \quad L[\phi] = \phi'' + p\phi' + q\phi.$$

Note that $L[\phi]$ is a function on I . The value of $L[\phi]$ at a point t is

$$(5.2) \quad L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t).$$

In this lecture we study the second order linear homogeneous equation $L[\phi](t) = 0$. Since $y = \phi(t)$, we will usually write this equation in the form

$$(5.3) \quad L[y] = y'' + p(t)y' + q(t)y = 0.$$

With Eq. (5.3) we associate a set of initial conditions

$$(5.4) \quad y(t_0) = y_0, \quad y'(t_0) = y'_0,$$

2 DIBYAJYOTI DEB, SECOND ORDER DIFFERENTIAL EQUATIONS

where t_0 is any point in the interval I , and y_0 and y'_0 are given real numbers.

The questions that we would like to ask include,

1. Does the initial value problem (5.3), (5.4) always have a solution.
2. If it has a solution then, does it have more than one.
3. Can anything be said about the form and structure of the solutions.

The first two questions are answered with the following theorem.

Theorem 5.5. (*Existence and Uniqueness Theorem*)

Consider the initial value problem

$$(5.6) \quad y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0,$$

where p, q , and g are continuous on an open interval I that contains the point t_0 . Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I .

The theorem says three things:

- The initial value problem *has* a solution; in other words; a solution *exists*.
- The initial value problem has *only one* solution; that is the solution is *unique*.
- The solution ϕ is defined *throughout the interval I* where the coefficients are continuous and is at least twice differentiable there.

We see an application of the above theorem in the next example.

Example 1. Determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

$$(5.7) \quad (t-1)y'' - 3ty' + 4y = \sin t, \quad y(-2) = 2, \quad y'(-2) = 1$$

Solution 1. If the given differential equation is written in the form of Eq. (5.6), then $p(t) = -3t/(t-1)$, $q(t) = 4/(t-1)$, and $g(t) = \sin t/(t-1)$. The only point of discontinuity of the coefficient is $t = 1$. Therefore, the longest open interval, containing the initial point $t = -2$, in which all the coefficients are continuous is $-\infty < t < 1$. Therefore, this is the longest interval in which the above theorem guarantees that the solution exists.

We look into this next theorem, which provides a way of finding more solutions, starting from two.

5.2. Solution of Linear Homogeneous Equations; the Wronskian 3

Theorem 5.8. (*Principle of Superposition*)

If y_1 and y_2 are two solutions of the differential equation,

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2 .

Now to answer our third question regarding the form and structure of the solutions of Eq. (5.3), we begin by examining whether the constants c_1 and c_2 from the theorem can be chosen so as to satisfy the initial conditions (5.4). These initial conditions require c_1 and c_2 to satisfy the equations

$$(5.9) \quad c_1y_1(t_0) + c_2y_2(t_0) = y_0,$$

$$(5.10) \quad c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'.$$

The determinant of the coefficients of the above system is

$$(5.11) \quad W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0).$$

If $W \neq 0$, then Eqs. (5.9), (5.10) have a unique solution (c_1, c_2) regardless of the values of y_0 and y_0' . On the other hand, if $W = 0$, then the same equations have no solution unless y_0 and y_0' satisfy a certain additional condition; in this case there are infinitely many solutions.

The determinant W is called the **Wronskian determinant**, or simply the **Wronskian**, of the solutions y_1 and y_2 . We use the next theorem for this new result.

Theorem 5.12. *Suppose that y_1 and y_2 are two solutions of Eq. (5.3)*

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the initial conditions (5.4)

$$y(t_0) = y_0, y'(t_0) = y_0'$$

are assigned. Then it is always possible to choose the constants c_1, c_2 so that

$$y = c_1y_1(t) + c_2y_2(t)$$

satisfies the differential equation (5.3) and the initial conditions (5.4) if and only if the Wronskian

$$W = y_1y_2' - y_1'y_2$$

is not zero at t_0 .

4 DIBYAJYOTI DEB, SECOND ORDER DIFFERENTIAL EQUATIONS

The previous theorem gives us a way of constructing infinite number of solutions starting from two solutions y_1 and y_2 , whose Wronskian is not zero at the initial point t_0 . The next theorem finally answers our third question about the form and structure of the solution of Eq. (5.3).

Theorem 5.13. *Suppose that y_1 and y_2 are two solutions of Eq. (5.3)*

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

Then the family of solutions

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of Eq. (5.3) if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero.

Theorem 5.13 states that, if and only if the Wronskian of y_1 and y_2 is not everywhere zero, then the linear combination $c_1y_1 + c_2y_2$ contains all solutions of Eq. (5.3). It is therefore natural to call the expression

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary constant coefficients the **general solution** of Eq. (5.3). The solutions y_1 and y_2 are said to form a **fundamental set of solutions** of Eq. (5.3) if and only if their Wronskian is nonzero.

We look at an application of the above theorem in the next example.

Example 2. Show that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are fundamental solutions of the differential equation

$$(5.14) \quad t^2y'' - 2y = 0$$

for $t > 0$.

Solution 2. We can verify that y_1 and y_2 are indeed solutions to Eq. (5.14) by substitution. To check whether they form a pair of fundamental solutions, we find the Wronskian,

$$(5.15) \quad W = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -1/t^2 \end{vmatrix} = t^2 \cdot \frac{-1}{t^2} - 2t \cdot t^{-1} = -3 \neq 0.$$

Since $W \neq 0$, therefore y_1 and y_2 form a fundamental set of solutions and therefore every other solution is of the form $c_1y_1 + c_2y_2$ for arbitrary constants c_1 and c_2 .

A new question that arises now, is whether a differential equation of the form (5.3) always has a fundamental set of solutions. The following theorem provides an affirmative answer to this question.

5.2. Solution of Linear Homogeneous Equations; the Wronskian 5

Theorem 5.16. Consider the differential equation (5.3)

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

whose coefficients p and q are continuous on some open interval I . Choose some point t_0 in I . Let y_1 be the solution of Eq. (5.3) that also satisfies the initial conditions

$$y(t_0) = 1, \quad y'(t_0) = 0,$$

and let y_2 be the solution of Eq. (5.3) that satisfies the initial conditions

$$y(t_0) = 0, \quad y'(t_0) = 1,$$

Then y_1 and y_2 form a fundamental solutions of Eq. (5.3).

The above theorem assures that a fundamental set of solutions always exists. In fact, a differential equation has infinitely many fundamental solutions.

Now let us examine further the properties of the Wronskian of two solutions of a second order linear homogeneous differential equation. The following theorem, gives a simple explicit formula for the Wronskian of any two solutions of any such equation, even if the solutions themselves are not known.

Theorem 5.17. (Abel' Theorem)

If y_1 and y_2 are solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on some open interval I , then the Wronskian $W(y_1, y_2)(t)$ is given by

$$(5.18) \quad W(y_1, y_2)(t) = ce^{-\int p(t) dt},$$

where c is a certain constant that depends on y_1 and y_2 , but not on t . Further, $W(y_1, y_2)(t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

The above theorem says that the Wronskian of any two fundamental sets of solutions of the same differential equation can differ only by a multiplicative constant, and that the Wronskian of any fundamental set of solutions can be determined, up to a multiplicative constant, without solving the differential equation.

We apply the above the theorem in the next example.

Example 3. Find the general form of the Wronskian of the equation

$$(5.19) \quad 2t^2y'' + 3ty' - y = 0, \quad t > 0$$

Solution 3. We write the differential equation in the standard form with the coefficient of y'' equal to 1. Thus we obtain,

$$(5.20) \quad y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0,$$

so $p(t) = 3/2t$. Hence

$$(5.21) \quad W(y_1, y_2)(t) = ce^{-\int \frac{3}{2t} dt} = ce^{-\frac{3}{2} \ln t} = ct^{-3/2}.$$

Equation (5.21) gives the Wronskian of any pair of solutions of the differential equation.

5.3. Complex roots of the Characteristic Equation

In the previous lecture we learned how to solve second order linear homogeneous equation with constant coefficients, whose characteristic equation has different real roots.

In this section we look into the same equation

$$(5.22) \quad ay'' + by' + cy = 0.$$

whose characteristic equation

$$(5.23) \quad ar^2 + br + c = 0.$$

has complex roots. Since the roots are conjugate complex numbers, we denote them by

$$(5.24) \quad r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu,$$

where λ and μ are real. The corresponding expressions for the two solutions are given by (Note the two solutions of equation (5.22) is given by $e^{r_1 t}$ and $e^{r_2 t}$.)

$$(5.25) \quad y_1(t) = e^{(\lambda+i\mu)t}, \quad y_2(t) = e^{(\lambda-i\mu)t}.$$

y_1 and y_2 can also be written as

$$(5.26) \quad y_1(t) = e^{\lambda t} e^{i\mu t}, \quad y_2(t) = e^{\lambda t} e^{-i\mu t},$$

We would like to see what it means to raise e to a complex power. The answer is provided by an important relation known as Euler's formula.

Euler's Formula. $e^{i\theta} = \cos \theta + i \sin \theta$.

Using Euler's formula we have $y_1(t) = e^{\lambda t}(\cos \mu t + i \sin \mu t)$, and $y_2(t) = e^{\lambda t}(\cos \mu t - i \sin \mu t)$.

However, rather than using the complex-valued solutions $y_1(t)$ and $y_2(t)$, let us seek instead a fundamental set of solutions of Eq. (5.22) that are real-valued. We know that any linear combination of two

solutions is also a solution, so let us form the linear combinations $y_1(t) + y_2(t)$ and $y_1(t) - y_2(t)$. In this way we obtain

$$(5.27) \quad y_1(t) + y_2(t) = 2e^{\lambda t} \cos \mu t, \quad y_1(t) - y_2(t) = 2ie^{\lambda t} \sin \mu t.$$

Dropping the multiplicative constants 2 and $2i$ for convenience, we are left with

$$(5.28) \quad u(t) = e^{\lambda t} \cos \mu t, \quad v(t) = e^{\lambda t} \sin \mu t.$$

$u(t)$ and $v(t)$ form a fundamental set of solutions since $W(u, v) = \mu e^{2\lambda t} \neq 0$ (since $\mu \neq 0$). Therefore the general solution of Eq. (5.22) is

$$(5.29) \quad y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t,$$

where c_1 and c_2 are arbitrary constants. We look into the next example which uses these results.

Example 4. Solve the given initial value problem.

$$(5.30) \quad y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Solution 4. The characteristic equation is $r^2 + 4r + 5 = 0$ and its roots are $r = -2 \pm i$. Thus the general solution of the differential equation is

$$(5.31) \quad y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t.$$

To apply the initial condition we set $t = 0$ in the the above equation; this gives

$$(5.32) \quad y(0) = c_1 = 1.$$

For the second initial condition we must differentiate Eq. (5.31) and then set $t = 0$.

$$(5.33) \quad y' = -2c_1 e^{-2t} \cos t - c_1 e^{-2t} \sin t - 2c_2 e^{-2t} \sin t + c_2 e^{-2t} \cos t.$$

$$(5.34) \quad y'(0) = -2c_1 + c_2 = 0.$$

Substituting $c_1 = 1$, we get $c_2 = 2$. Using these values of c_1 and c_2 in Eq. (5.31), we obtain

$$(5.35) \quad y = e^{-2t} \cos t + 2e^{-2t} \sin t.$$

as the solution of the initial value problem (5.30).

A good question to ask now, is how do the graph of the solution look like. The presence of trigonometric factors in the solution makes the graph into an oscillation. The exponential factor determines the nature of the oscillation as follows.

- If $\lambda > 0$, then the oscillations increase with time.
- If $\lambda < 0$, then the oscillations decrease with time.

8 DIBYAJYOTI DEB, SECOND ORDER DIFFERENTIAL EQUATIONS

- If $\lambda = 0$, then the oscillations stays constant with time.

Since $\lambda = -2$ in the previous example, therefore the oscillations decay with time.