First and Second Order Differential Equations Lecture 4

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4.1. Outline of Lecture

- The Existence and the Uniqueness Theorem
- Homogeneous Equations with Constant Coefficients

4.2. The Existence and the Uniqueness Theorem

We have looked at the existence and uniqueness theorem for nonlinear equations in the previous lecture. However verifying the theorem especially for nonlinear equations require solving the initial value problem. In general, finding a solution is not feasible because there is no method of solving the differential equation that applies in all cases.

Therefore for the general case, it is necessary to adopt an indirect approach that demonstrates the existence of a solution. The heart of this method is the construction of a sequence of functions that converges to a limit function satisfying the initial value problem, although the members of the sequence individually do not.

We note that it is sufficient to consider the problem in which the initial point is the origin; that is we consider the problem

$$y' = f(t, y), \qquad y(0) = 0. \tag{4.1}$$

If some other initial point is given, then we can always make a preliminary change of variables, corresponding to a translation of the coordinate axes, that will take the given point to the origin. We can thus modify the existence and uniqueness theorem in the following way.

Theorem 4.2. *If f and $\partial f/\partial y$ are continuous in a rectangle $R : |t| \leq a, |y| \leq b$, then there is some interval $|t| \leq h \leq a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem (4.1).*

For the method of proof, it is necessary to transform the initial value problem (4.1) into a more convenient form. If we suppose temporarily that there is a differentiable function $y = \phi(t)$ that satisfies the initial value problem, then $f[t, \phi(t)]$ is a continuous function of t only. Hence we can integrate $y' = f(t, y)$ from the initial point $t = 0$ to an arbitrary value of t, obtaining

$$\phi(t) = \int_0^t f[s, \phi(s)]\, ds \tag{4.3}$$

where we have made use of the initial condition $\phi(0) = 0$. The above equation is called an **integral equation**.

One method of showing that the integral equation (4.3) has a unique solution is known as the **method of successive approximations** or Picard's **iteration method**.

We start by choosing an initial function ϕ_0. The simplest choice is

$$\phi_0(t) = 0 \tag{4.4}$$

then ϕ_0 at least satisfies the initial condition in Eq. (4.1), although presumable not the differential equation. The next approximation ϕ_1 is obtained by substituting $\phi_0(s)$ for $\phi(s)$ in the right side of Eq. (4.3) and calling the result of this operation $\phi_1(t)$. Thus

$$\phi_1(t) = \int_0^t f[s, \phi_0(s)]\, ds. \tag{4.5}$$

Similarly, ϕ_2 is obtained from ϕ_1,

$$\phi_2(t) = \int_0^t f[s, \phi_1(s)]\, ds. \tag{4.6}$$

and, in general,

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\, ds. \tag{4.7}$$

In this manner we generate a sequence of functions $\{\phi_n\} = \phi_0, \phi_1, \ldots, \phi_n, \ldots$. We look into this infinite sequence in the next example.

Example 1. Solve the initial value problem

$$y' = ty + 1, \qquad y(0) = 0 \tag{4.8}$$

by the method of successive approximations.

Solution 1. If $y = \phi(t)$ is the solution then the corresponding integral equation is

$$\phi(t) = \int_0^t (s\phi(s) + 1)\, ds \tag{4.9}$$

If the initial approximation is $\phi_0(t) = 0$, it follows that

$$\phi_1(t) = \int_0^t (s\phi_0(s) + 1)\, ds = \int_0^t ds = t. \tag{4.10}$$

Similarly,

$$\phi_2(t) = \int_0^t (s\phi_1(s) + 1)\, ds = \int_0^t (s^2 + 1)\, ds = \frac{t^3}{3} + t. \tag{4.11}$$

and

$$\phi_3(t) = \int_0^t (s\phi_2(s) + 1)\, ds = \int_0^t (\frac{s^4}{3} + s^2 + 1)\, ds = \frac{t^5}{3 \cdot 5} + \frac{t^3}{3} + t. \tag{4.12}$$

Equations (4.10), (4.11), and (4.12) suggest that

$$\phi_n(t) = t + \frac{t^3}{3} + \frac{t^5}{3 \cdot 5} + \ldots + \frac{t^{2n-1}}{3 \cdot 5 \cdots (2n-1)} \tag{4.13}$$

for each $n \geq 1$, and this result can be established by mathematical induction (Try it!).

It follows from Eq. (4.13) that $\phi_n(t)$ is the nth partial sum of the infinite series

$$\sum_{k=1}^{\infty} \frac{t^{2k-1}}{3 \cdot 5 \cdots (2k-1)} \tag{4.14}$$

hence $\lim_{n\to\infty} \phi_n(t)$ exists if and only if the series (4.14) converges. Applying the ratio test, we see that, for each t,

$$\left| \frac{t^{2k+1} \cdot 3 \cdot 5 \cdots (2k-1)}{3 \cdot 5 \cdots (2k+1) \cdot t^{2k-1}} \right| = \frac{t^2}{2k+1} \to 0 \text{ as } k \to \infty \tag{4.15}$$

Thus the series (4.14) converges for all t, and its sum $\phi(t)$ is the limit of the sequence $\{\phi_n(t)\}$. We can verify by direct substitution that $\phi(t) = \sum_{k=1}^{\infty} \frac{t^{2k-1}}{3 \cdot 5 \cdots (2k-1)}$ is a solution of the integral equation (4.14).

4.3. Homogeneous Equations with Constant Coefficients

We now shift our focus to second order equations. A second order ordinary differential equation has the form

$$\frac{d^2 y}{dt^2} = f(t, y, \frac{dy}{dt}), \tag{4.16}$$

where f is some given function. Equation (4.16) is said to be **linear** if the function f has the form

$$f(t, y, \frac{dy}{dt}) = g(t) - p(t)\frac{dy}{dt} - q(t)y, \tag{4.17}$$

where g, p, and q are specified functions of the independent variable t but do not depend on y. In this case we usually rewrite Eq. (4.16) as

$$y'' + p(t)y' + q(t)y = g(t), \tag{4.18}$$

Instead of Eq. (4.18), we often see the equation

$$P(t)y'' + Q(t)y' + R(t)y = G(t). \tag{4.19}$$

If $P(t) \neq 0$, we can divide Eq. (4.19) by $P(t)$ and thereby obtain Eq. (4.18) with

$$p(t) = \frac{Q(t)}{P(t)}, \qquad q(t) = \frac{R(t)}{P(t)}, \qquad g(t) = \frac{G(t)}{P(t)}. \tag{4.20}$$

If Eq. (4.16) is not of the form (4.18) or (4.19), then it is called **nonlinear**.

An initial value problem consists of a differential equation such as Eq. (4.16), or (4.18) together with a pair of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y'_0 \tag{4.21}$$

where y_0 and y'_0 are given numbers prescribing values for y and y' at the initial point t_0. Since we have a second order differential equation therefore, roughly speaking two integrations are required to find a solution and each integration introduces an arbitrary constant. Hence we have two initial conditions.

A second order linear equation is said to be **homogeneous** if the term $g(t)$ in Eq. (4.18) is zero for all t. Otherwise the equation is called **nonhomogeneous**. Therefore a homogeneous equation is of the form

$$y'' + p(t)y' + q(t)y = 0 \tag{4.22}$$

In this section we will concentrate our attention on equations in which the functions P, Q and R are constants. In this case Eq. (4.19) becomes

$$ay'' + by' + cy = 0 \tag{4.23}$$

where a, b and c are given constants.

We now see how we can solve the above equation. We start by seeking exponential solutions of the form $y = e^{rt}$, where r is a parameter to be determined. Then it follows that $y' = re^{rt}$ and $y'' = r^2e^{rt}$. By substituting these expressions for y, y', and y'' in Eq. (4.23) we obtain

$$(ar^2 + br + c)e^{rt} = 0 \tag{4.24}$$

Since $e^{rt} \neq 0$,

$$ar^2 + br + c = 0 \tag{4.25}$$

Equation (4.25) is called the **characteristic equation** for the differential equation (4.23). Solving this quadratic equation gives us two roots r_1 and r_2. In this section we consider the case when both r_1 and r_2 are real and $r_1 \neq r_2$. Then the two solutions are $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$. Therefore

$$y = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{4.26}$$

is also a solution of Eq. (4.23) for arbitrary constants c_1 and c_2. We look into an example below which illustrates the method.

Example 2. Find the solution of the initial value problem

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \qquad y'(0) = 3. \tag{4.27}$$

Solution 2. We assume that $y = e^{rt}$, and it then follows that r must be a root of the characteristic equation

$$r^2 + 5r + 6 = (r+2)(r+3) = 0. \tag{4.28}$$

Thus the possible values of r are $r_1 = -2$ and $r_2 = -3$; the general solution of Eq. (4.27) is

$$y = c_1 e^{-2t} + c_2 e^{-3t}. \tag{4.29}$$

To satisfy the first initial condition, we set $t = 0$ and $y = 2$ in Eq. (4.29); thus c_1 and c_2 must satisfy

$$c_1 + c_2 = 2. \tag{4.30}$$

To use the second initial condition, we must first differentiate Eq. (4.29). This gives $y' = -2c_1 e^{-2t} - 3c_2 e^{-3t}$. Then, setting $t = 0$ and $y' = 3$, we obtain

$$-2c_1 - 3c_2 = 3. \tag{4.31}$$

By solving Eqs. (4.30) and (4.31), we find that $c_1 = 9$ and $c_2 = -7$. Therefore the solution of the initial value problem (4.27) is

$$y = 9e^{-2t} - 7e^{-3t} \tag{4.32}$$

Note that as $t \to \infty$, the solution $y \to 0$. In general as t increases, the magnitude of the solution either tends to zero (when both exponents are negative) or else grows rapidly (when at least one exponent is positive).