First Order Differential Equations
Lecture 3

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3.1. Outline of Lecture

- Differences Between Linear and Nonlinear Equations
- Exact Equations and Integrating Factors

3.2. Differences between Linear and Nonlinear Equations

We have looked at first order equations so far, both linear and nonlinear. We have developed methods of solving linear equations and some subclasses of nonlinear equations. We now discuss some important ways in which nonlinear equations differ from linear ones.

- **Existence and Uniqueness of Solutions.** So far, we have discusses a number of initial value problems, each of which had a solution and apparently only one solution. This raises the question whether every initial value problem has exactly one solution. The answer to this question is given by the following theorem.

  Theorem 3.1. *If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation*

$$\frac{dy}{dt} + p(t)y = g(t) \tag{3.2}$$

  for each t in I, and that also satisfies the initial condition has a unique solution.

$$y(t_0) = y_0 \tag{3.3}$$

where y_0 is an arbitrary prescribed initial value.

Note that Theorem 3.1 states that the given initial value problem *has* a solution and also that the problem has *only one* solution. In other words, the theorem asserts both the *existence* and *uniqueness* of the solution of the solution of the initial value problem.

We apply this theorem in the next example.

Example 1. Find an interval in which the initial value problem

$$(t-3)y' + (\ln t)y = 2t, \qquad y(1) = 2 \tag{3.4}$$

Solution 1. Rewriting the above equation in the standard form, we have

$$y' + \frac{\ln t}{t-3}y = \frac{2t}{t-3}$$

So $p(t) = \dfrac{\ln t}{t-3}$ and $g(t) = \dfrac{2t}{t-3}$. g is continuous for all $t \neq 3$. p is continuous for all $t \neq 0, 3$. Therefore p and g are both continuous on the interval $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$. The interval $(0, 3)$ contains the initial point $t = 1$. Therefore Theorem 3.1 guarantees that the problem has a unique solution on the interval $0 < t < 3$.

We now turn our attention to nonlinear differential equations and modify Theorem 3.1 by a more general theorem.

Theorem 3.5. *Let the function f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta, \gamma < t < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ containing in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem*

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{3.6}$$

This is a more general theorem since it reduces to Theorem 3.1 if the differential equation is linear. For then $f(t, y) = -p(t)y + g(t)$ and $\partial f(t, y)/\partial y = -p(t)$, so the continuity of f and $\partial f/\partial y$ is equivalent to the continuity of p and g in this case.

Note that the conditions stated in Theorem 3.5 are sufficient to guarantee the existence of a unique solution of the initial value problem (3.6) in some interval $t_0 - h < t < t_0 + h$, but they are not necessary. That is, the conclusion remains true under slightly weaker hypotheses about the function f. In fact, the existence of solution (but not its uniqueness) can be established on the basis of the continuity of f alone.

We look at an example below making use of the above theorem.

Example 2. Solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

$$y' = -\frac{4t}{y}, \qquad y(0) = y_0 \tag{3.7}$$

Solution 2. For this equation $f = -4t/y$ and $\partial f/\partial y = 4t/y^2$. f and $\partial f/\partial y$ are continuous in any rectangle where $y \neq 0$. This is also a separable equation. Cross multiplication makes the equation separable for integration.

$$\int y\,dy = \int -4t\,dt$$

Integrating both sides we have,

$$\frac{y^2}{2} = -2t^2 + C \tag{3.8}$$

for some constant C. Using the initial values we have $C = y_0^2/2$. Using this value for C and simplifying both sides we have

$$y = \pm\sqrt{-4t^2 + y_0^2}$$

However we would like to find out the interval where the solution exists. The term inside the radical has to be non-negative. Therefore $4t^2 < y_0^2$ or $|t| < |y_0|/2$. By Theorem 3.5, we get the extra condition that $y_0 \neq 0$ (Since f and $\partial f/\partial y$ are continuous in any rectangle where $y \neq 0$ containing the point $(0, y_0)$).

Therefore the solution looks like

$$y = \pm\sqrt{-4t^2 + y_0^2}, \quad y_0 \neq 0, \quad |t| < |y_0|/2. \tag{3.9}$$

Now let's look at some other differences between linear and nonlinear equations.

- **Interval of Definition.** According to Theorem 3.1, the solution to a linear equation (3.2) subject to the initial condition $y(t_0) = y_0$, exists throughout any interval about $t = t_0$ in which the functions p and g are continuous.

  On the other hand, for a nonlinear initial value problem satisfying the hypotheses of Theorem 3.5, the interval in which a solution exists may be difficult to determine. This is because it is not so easy to determine the solution $y = \phi(t)$ of a nonlinear equation unlike a linear equation.

- **General Solution.** Another way in which linear and nonlinear equations differ concerns the concept of general solution.

  For a first order linear equation it is possible to obtain a solution containing one arbitrary constant, from which all possible solutions follow by specifying values for this constant as we have seen in the previous lecture.

  For nonlinear equations this is not the case; even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by giving values to this constant.

- **Implicit Solutions.** The solution for an initial value problem of a first order linear equation provides an *explicit* formula for the solution $y = \phi(t)$.

  However for a nonlinear equation, the solution is *implicit* in nature, of the form $F(t, y) = 0$.

3.3. Exact Equations and Integrating Factors

In this section we look at a different class of nonlinear equations known as **exact** equations for which there is also a well-defined method of solution. We define an exact equation in the next theorem along with another result.

Theorem 3.10. *Let the functions M, N, M_y, and N_x, where subscripts denote partial derivatives, be continuous in the rectangular region R : $\alpha < x < \beta, \gamma < y < \delta$. Then*

$$M(x, y) + N(x, y)y' = 0 \tag{3.11}$$

is an **exact** *differential equation in R if and only if*

$$M_y(x, y) = N_x(x, y) \tag{3.12}$$

at each point of R. That is, there exists a function ψ satisfying

$$\psi_x(x, y) = M(x, y), \qquad \psi_y(x, y) = N(x, y), \tag{3.13}$$

if and only if M and N satisfy Eq. (3.12).

To find the expression for the solution to the equation (3.11) we see that,

$$M(x, y) + N(x, y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi(x, y) \tag{3.14}$$

Therefore Eq. (3.11) becomes

$$\frac{d}{dx}\psi(x,y) = 0. \tag{3.15}$$

Hence the solution to Eq. (3.11) is given implicitly by

$$\psi(x,y) = C. \tag{3.16}$$

for an arbitrary constant C. Let's illustrate the above theorem in the next example.

Example 3. Determine whether the equation is exact. If it is exact, find the solution.

$$(3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0. \tag{3.17}$$

Solution 3. $M(x,y) = 3x^2 - 2xy + 2$ and $N(x,y) = 6y^2 - x^2 + 3$. Therefore $M_y(x,y) = -2x$ and $N_x(x,y) = -2x$. Since they are the same, hence Eq. (3.17) is exact. Thus there is a $\psi(x,y)$ such that

$$\psi_x(x,y) = 3x^2 - 2xy + 2.$$

$$\psi_y(x,y) = 6y^2 - x^2 + 3.$$

Integrating the first of these equations with respect to x, we obtain

$$\psi(x,y) = x^3 - x^2y + 2x + h(y). \tag{3.18}$$

Differentiating the above equation with respect to y, we obtain,

$$\psi_y(x,y) = -x^2 + h'(y).$$

Setting $\psi_y = N$ gives

$$-x^2 + h'(y) = 6y^2 - x^2 + 3.$$

Thus $h'(y) = 6y^2 + 3$ and $h(y) = 2y^3 + 3y$. The constant of integration can be omitted since any solution of the preceding equation is satisfactory. Substituting for $h(y)$ in Eq. (3.18) gives

$$\psi(x,y) = x^3 - x^2y + 2x + 6y^2 - x^2 + 3. \tag{3.19}$$

Hence solutions of Eq. (3.17) are given implicitly by

$$x^3 - x^2y + 2x + 6y^2 - x^2 + 3 = C. \tag{3.20}$$

A valid question to ask now is what happens when the initial equation isn't exact. In that situation it is sometimes possible to convert it into an exact equation by multiplying the equation by a suitable integrating factor $\mu(x,y)$.

Unfortunately even though integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The most important situations in which simple integrating factors can be found occur when μ is a function of only one of the variables x or y, instead of both.

If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor μ that also depends on x and it satisfies the differential equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \tag{3.21}$$

If $(N_x - M_y)/M$ is a function of y only, then there is an integrating factor μ that also depends on y and it satisfies the differential equation

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu \tag{3.22}$$

Finally we look into an example in which the equation is not exact to begin with but is made exact by multiplying with an integrating factor.

Example 4. Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0. \tag{3.23}$$

and then solve the equation.

Solution 4. Here $M(x,y) = 3xy + y^2$ and $N(x,y) = x^2 + xy$. $M_y \neq N_x$, therefore the differential equation isn't exact. We compute $(M_y - N_x)/N$ and find that

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}.$$

Thus there is an integrating factor μ that is a function of x only, and it satisfies the differential equation

$$\frac{d\mu}{dx} = \frac{\mu}{x} \tag{3.24}$$

Solving the above differential equation we have $\mu(x) = x$. Multiplying Eq. (3.23) by this integrating factor, we obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0. \tag{3.25}$$

This equation is exact. Therefore there exists a function $\psi(x,y)$, such that

$$\psi_x(x,y) = 3x^2y + xy^2.$$
$$\psi_y(x,y) = x^3 + x^2y.$$

Integrating the first of these equations with respect to x, we obtain

$$\psi(x,y) = x^3y + \frac{x^2y^2}{2} + h(y). \tag{3.26}$$

Differentiating the above equation with respect to y, we have

$$\psi_y(x, y) = x^3 + x^2 y + h'(y).$$

Setting $\psi_y = N$ gives

$$x^3 + x^2 y + h'(y) = x^3 + x^2 y.$$

Therefore $h'(y) = 0$, hence $h(y) = C = 0$. We can choose this constant to be zero since any solution of the preceding equation is satisfactory. Substituting for $h(y)$ in Eq. (3.26) gives

$$\psi(x, y) = x^3 y + \frac{x^2 y^2}{2}. \tag{3.27}$$

Hence solutions of Eq. (3.23) are given implicitly by

$$x^3 y + \frac{x^2 y^2}{2} = C. \tag{3.28}$$