First Order Differential Equations
Lecture 2

Dibyajyoti Deb

2.1. Outline of Lecture

- Linear Equations; Method of Integrating Factors
- Separable Equations
- Modeling with First Order Equations

2.2. Linear Equations; Method of Integrating Factors

The most general first order equation is of the form

$$\frac{dy}{dt} = f(t, y) \tag{2.1}$$

where f is a given function of two variables. Any differential function $y = \phi(t)$ that satisfies this equation for all t in some interval is called a solution, and the object is to determine whether such functions exist and, if so, to develop methods for finding them. Unfortunately, for an arbitrary function f, there is no general method of solving the equation in terms of elementary functions.

If the function f in Eq. (2.1) depends linearly on the dependant variable y, then Eq. (2.1) is called a first order linear equation.

The general **first order linear equation** is of the form

$$\frac{dy}{dt} + p(t)y = g(t), \tag{2.2}$$

where both $p(t)$ and $g(t)$ are continuous functions.

The method described in the previous lecture to solve the differential equation describing the motion of the falling object doesn't work here. So we need a different method of solution for it. It involves multiplying the differential equation (2.2) by a certain function $\mu(t)$, chosen so that the resulting equation is readily integrable. The function $\mu(t)$ is called an **integrating factor**.

We look into the method of finding the integrating factor briefly. We multiply Eq. (2.2) by $\mu(t)$, obtaining

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \tag{2.3}$$

We see that the left side of Eq. (2.3) is the derivative of the product $\mu(t)y$ if we assume that $\mu(t)$ will satisfy the following

$$\frac{d\mu(t)}{dt} = p(t)\mu(t) \tag{2.4}$$

We have

$$\frac{d\mu(t)}{\mu(t)} = p(t)\,dt$$

and consequently

$$\ln\mu(t) = \int p(t)\,dt + C.$$

after integrating both sides. By choosing the arbitrary constant C to be zero, we obtain the simplest possible function for μ, namely,

$$\mu(t) = e^{\int p(t)\,dt}. \tag{2.5}$$

An example is shown below which uses the method of integrating factors to solve a first order linear equation.

Example 1. Solve the initial value problem.

$$ty' + 2y = t^2 - t + 1, \qquad y(1) = \frac{1}{2}, t > 0$$

Solution 1. We bring the original equation to the form (2.2) by dividing both sides of the equation by t.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t} \tag{2.6}$$

The integrating factor is $\mu(t) = e^{\int \frac{2}{t}\,dt} = t^2$. Multiplying both sides of the above equation by t^2 we get

$$t^2y' + 2ty = t^3 - t^2 + t \tag{2.7}$$

The left side of the equation is the derivative of the product t^2y. Therefore

$$\frac{d(t^2y)}{dt} = t^3 - t^2 + t \tag{2.8}$$

Multiplying both sides by dt and then integrating both sides, we have

$$t^2 y = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C \tag{2.9}$$

To find C we use the initial value condition to get

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C$$

Therefore $C = \dfrac{1}{12}$.

Using this value of C in Eq. (2.9), and then dividing by t^2 we have the solution to the initial value problem.

$$y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2} \tag{2.10}$$

2.3. Separable Equations

In this section we look into a subclass of nonlinear equations that can be solved by direct integration. We can rewrite the general first order equation (2.1) in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{2.11}$$

If it happens that M is a function of x only and N is a function of y only, then the above equation becomes

$$M(x) + N(y)\frac{dy}{dx} = 0 \tag{2.12}$$

Such an equation is said to be **separable**, because it can be written in the differential form

$$M(x)\,dx + N(y)\,dy = 0 \tag{2.13}$$

We can solve the above the equation by integrating the functions M and N. Usually this results in an implicit solution. We illustrate this in the following example.

Example 2. Solve the equation

$$\frac{dy}{dx} = \frac{x^3}{1 - y^2}$$

Solution 2. Cross multiplication makes the equation separable for integration

$$\int (1 - y^2)\,dy = \int x^3\,dx \tag{2.14}$$

Integrating both sides(and bringing the constants on one side) we get

$$y - \frac{y^3}{3} = \frac{x^4}{4} + C$$

Therefore

$$y - \frac{y^3}{3} - \frac{x^4}{4} = C \tag{2.15}$$

where C is an arbitrary constant. Note that the final general solution (2.15) is implicit.

2.4. Modeling with First Order Equations

Now that we have a general idea of first order equations, we can use them to investigate a wide variety of problems in the physical, biological and social sciences. There are three basic steps in solving a problem using differential equation.

- Construction of the model - In this step, translate the physical situation into mathematical terms. The differential equation is a mathematical model of the process.
- Analysis of the model - Once the problem has been formulated mathematically, it's time to solve the one or more differential equations involved with the model (or atleast finding out as much as possible about the properties of the solution).
- Comparison with Experiment or Observation - Finally, having obtained the solution (or atleast some information about it), interpret this information in the context in which the problem arose.

We look at an example below.

Example 3. A tank initially contains 120 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 3 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

Solution 3. Since the incoming and outgoing flows of water are the same, the amount of water in the tank remains constant at 120 L. Let the amount of salt in the tank at any time t be denoted by $Q(t)$. Thus

$$\frac{dQ}{dt} = \text{rate in - rate out} \tag{2.16}$$

where "rate in" and "rate out" refers to the rate at which the salt flows into and out of the tank, respectively.

$$\text{rate in} = \gamma \text{ g/L} \times 3 \text{ L/min} = 3\gamma \text{ g/min}. \tag{2.17}$$

The concentration of salt in the tank at any time t is $\frac{Q(t)}{120}$ g/L, thus

$$\text{rate out} = \frac{Q(t)}{120} \text{ g/L} \times 3 \text{ L/min} = \frac{Q(t)}{40} \text{ g/min}. \tag{2.18}$$

To make it convenient we omit the units during our calculation. Therefore,

$$\frac{dQ}{dt} = 3\gamma - \frac{Q(t)}{40} \tag{2.19}$$
$$\frac{dQ}{dt} = \frac{120\gamma - Q(t)}{40}$$

This is a first order separable equation (Note that (2.19) is also a first order linear equation).

Cross multiplication makes the equation separable for integration.

$$\int \frac{dQ}{Q(t) - 120\gamma} = \int -\frac{dt}{40} \tag{2.20}$$

Integrating both sides we have,

$$\ln|Q(t) - 120\gamma| = -\frac{t}{40} + C \tag{2.21}$$

To find C, note that the tank initially contains pure water, therefore $Q(0) = 0$. Using this in the above equation we have $C = \ln|120\gamma|$. Putting this value of C back into the above solution and simplifying we have,

$$Q(t) = 120\gamma + |120\gamma|e^{-t/40} \tag{2.22}$$

The initial condition is true if $|120\gamma| = -120\gamma$. Therefore the final solution is

$$Q(t) = 120\gamma - 120\gamma e^{-t/40} \tag{2.23}$$

To find the limiting amount of salt as $t \to \infty$, we find

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}(120\gamma - 120\gamma e^{-t/40}) = 120\gamma.$$

This means that after a very long time the amount of salt in the tank will be 120γ g.