Series Solutions of Second Order Linear Equations Lecture 13

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13.1. Outline of Lecture

- Series Solutions near an Ordinary Point, Part II.
- Euler Equations.

13.2. Series Solutions near an Ordinary Point, Part II.

In the previous lecture, we considered the problem of finding solutions of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{13.1}$$

where P, Q, and R are polynomials, in the neighborhood of an ordinary point x_0. Assuming that Eq. (13.1) does have a solution $y = \phi(x)$ and that ϕ has a Taylor series

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \tag{13.2}$$

which converges for $|x - x_0| < \rho$, where $\rho > 0$, we found a_n can be determined by directly substituting the series (13.2) for y in Eq. (13.1).

We now consider how we might justify the statement that if x_0 is an ordinary point of Eq. (13.1) then there exists solutions of the form (13.2).

Suppose there is a solution of Eq. (13.1) of the form (13.2). By differentiating Eq. (13.2) m times and setting x equal to x_0 we have

$$m!a_m = \phi^{(m)}(x_0). \tag{13.3}$$

Hence, to compute a_n from the above expression, we need to determine $\phi^{(n)}(x_0)$ for $n = 0, 1, 2, \ldots$.

To compute $\phi^{(n)}(x_0)$, we use the original differential equation (13.1). Since ϕ is a solution of Eq. (13.1), we have

$$P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0. \tag{13.4}$$

We can find $\phi''(x)$ from the above equation

$$\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x), \tag{13.5}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. Setting x equal to x_0 in Eq. (13.5) gives

$$\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0). \tag{13.6}$$

From here we can find a_2 since

$$2!a_2 = \phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0). \tag{13.7}$$

It can be easily checked that $\phi'(x_0) = a_1$ and $\phi(x_0) = a_0$. Therefore

$$2!a_2 = \phi''(x_0) = -p(x_0)a_1 - q(x_0)a_0. \tag{13.8}$$

To determine a_3, we differentiate Eq. (13.5) and set x equal to x_0, obtaining

$$3!a_3 = \phi'''(x_0) = -2!p(x_0)a_2 - [p'(x_0) + q(x_0)]a_1 - q'(x_0)a_0. \tag{13.9}$$

As we see from above to compute the remaining a_n's we have to compute infinitely many derivatives of p and q. Unfortunately, this condition is too weak to ensure that we can prove the convergence of the resulting series expansion for $y = \phi(x)$. What is needed is to assume that the functions p and q are *analytic* at x_0.

With this we can generalize the definitions of an ordinary point and singular point of Eq. (13.1) as follows: if the functions $p = Q/P$ and $q = R/P$ are analytic at x_0, then the point x_0 is said to be an **ordinary point** of the differential equation (13.1); otherwise it is a **singular point**.

Now we shift our focus to finding the interval of convergence of the series solution. We look into a theorem which answers the question for a wide class of problems.

Theorem 13.10. *If x_0 is an ordinary point of the differential equation (13.1)*

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{13.11}$$

that is, if $p = Q/P$ and $q = R/P$ are analytic at x_0, then the general solution of Eq. (13.1) is

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 y_1(x) + a_1 y_2(x), \tag{13.12}$$

where a_0 and a_1 are arbitrary, and y_1 and y_2 are two power series solutions that are analytic at x_0. The solutions y_1 and y_2 form a fundamental set of solutions. Further, the radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

We will not prove this theorem, here however, there is an easier way to compute the lower bound of the radius of convergence of the series solution when P, Q and R are polynomials. We present it in the next two results.

13.2.1. Result 1

The ratio of two polynomials, say, Q/P, has a convergent power series expansion about a point $x = x_0$ if $P(x_0) \neq 0$.

13.2.2. Result 2

If any factors common to Q and P have been canceled, then the radius of convergence of the power series of Q/P about the point x_0 is precisely the distance from x_0 to the nearest root of P.

We use these results in the form of an example below.

Example 1. Determine a lower bound for the radius of convergence of the series solution about the given point x_0, for the given differential equation.

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0; \quad x_0 = 4. \tag{13.13}$$

Solution 1. The roots of $P(x) = x^2 - 2x - 3$ are 3 and -1. The nearest root to $x_0 = 4$ is the root 3 and the distance is 1. Hence the lower bound for the radius of convergence of the series solution of the differential equation is 1, i.e. the series solution $\sum_{n=0}^{\infty} a_n(x-4)^n$ converges for at least $|x - 4| < 1$.

13.3. Euler Equations.

In this section we will begin to consider how to solve equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{13.14}$$

in the neighborhood of a singular point x_0, i.e. where $P(x_0) = 0$. Instead of looking at any general equation, we will only consider a special type of second order equation called the Euler equation in this lecture.

13.3.1. Euler Equations.

A simple differential equation that has a singular point is the **Euler equation**

$$x^2y'' + \alpha xy' + \beta y = 0, \tag{13.15}$$

where α and β are real constants.

To solve a equation of this type our initial assumption of a solution would be $y = x^r$ for any constant r. Substituting back into Eq. (13.15) we have

$$x^2(x^r)'' + \alpha x(x^r)' + \beta x^r = 0.$$
$$x^r[r(r-1) + \alpha r + \beta] = 0.$$

We call the quadratic equation in r

$$r(r-1) + \alpha r + \beta = r^2 + (\alpha - 1)r + \beta = 0 \tag{13.16}$$

the characteristic equation. Based on the roots r_1 and r_2 of Eq. (13.16), we have the following solutions of Eq. (13.15).

- If r_1 and r_2 are real and $r_1 \neq r_2$, the general solution is

$$y = c_1|x|^{r_1} + c_2|x|^{r_2}. \tag{13.17}$$

- If r_1 and r_2 are real and $r_1 = r_2$, the general solution is

$$y = c_1|x|^{r_1} + c_2|x|^{r_1}\ln|x|. \tag{13.18}$$

- If r_1 and r_2 are complex then let $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, the general solution is

$$y = c_1|x|^{\lambda}\cos(\mu \ln|x|) + c_2|x|^{\lambda}\sin(\mu \ln|x|). \tag{13.19}$$

  for arbitrary constants c_1 and c_2 which can be determined with initial conditions.

We present an example of an Euler equation below.

Example 2. Determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$x^2y'' - xy' + y = 0 \tag{13.20}$$

Solution 2. For this Euler equation $\alpha = -1$ and $\beta = 1$. Hence the characteristic equation is

$$r^2 - 2r + 1 = 0 \tag{13.21}$$

whose root 1 is repeated. Hence the general solution is

$$y = c_1|x| + c_2|x|\ln|x|. \tag{13.22}$$

for arbitrary constants c_1 and c_2.