Series Solutions of Second Order Linear Equations Lecture 12

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12.1. Outline of Lecture

- Review of Power Series.
- Series Solutions near an Ordinary Point, Part I.

12.2. Review of Power Series.

Our goal from the very beginning has been to find the solution of a general second order equation without any restrictions to the coefficients or the forcing functions. In this regard we have given a systematic procedure for constructing solutions if the equation has constant coefficients. To deal with the much larger class of equations that have variable coefficients, it is necessary to extend our search for solutions beyond the familiar elementary functions of calculus. The principal tool that we need is the representation of a given function by a power series.

In this section we start by looking at some basic properties of power series.

12.2.1. Quick review of Power Series.

1. A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to converge at a point x if

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x-x_0)^n \tag{12.1}$$

exists for that x. The series certainly converges for $x = x_0$; it may converge for all x, or it may converge for some values of x and nor for others.

2. The series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to converge absolutely at a point x if the series

$$|\sum_{n=0}^{\infty} a_n(x-x_0)^n| = \sum_{n=0}^{\infty} |a_n||(x-x_0)^n| \tag{12.2}$$

converges. A thing to note is that absolute convergences implies convergence but not the other way around.

3. One of the most useful tests for the absolute convergence of a power series is the **ratio test**. If $a_n \neq 0$, and if, for a fixed value of x,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| = |x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x-x_0|L, \tag{12.3}$$

then the power series converges absolutely at that value of x if $|x-x_0|L < 1$ and diverges if $|x-x_0|L > 1$. If $|x-x_0|L = 1$, the test is inconclusive.

4. If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for $x = x_1$, it converges absolutely for $|x-x_0| < |x_1-x_0|$; and if it diverges at $x = x_1$, it diverges for $|x-x_0| > |x_1-x_0|$.

5. There is a nonnegative number ρ, called the **radius of convergence**, such that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely for $|x-x_0| < \rho$ and diverges for $|x-x_0| > \rho$. For a series that converges only at x_0, we define ρ to be zero; for a series that converges for all x, we say that ρ is infinite. If $\rho > 0$, then the interval $|x-x_0| < \rho$ is called the **interval of convergence**.

6. The value of a_n is given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}. \tag{12.4}$$

The series is called the Taylor series for the function f about $x = x_0$.

7. A function f that has a Taylor series expansion about $x = x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n, \tag{12.5}$$

with radius of convergence $\rho > 0$, is said to be **analytic** at $x = x_0$.

We look into couple of examples below.

Example 1. For which values of x does the power series

$$\sum_{n=1}^{\infty}(-1)^{n+1}n(x-3)^n \tag{12.6}$$

converge?

Solution 1. We use the ratio test. We have

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)(x-3)^{n+1}}{(-1)^{n+1}n(x-3)^n}\right| = |x-3|\lim_{n\to\infty}\frac{n+1}{n} = |x-3|. \tag{12.7}$$

According to statement 3, the series converges absolutely for $|x-3|<1$, or $2<x<4$, and diverges for $|x-3|>1$, or $x>4$ and $x<2$. To find what happens at $x=2$ and $x=4$ we substitute these values back into the original power series to see that both the series diverges since the nth term does not approach zero as $n\to\infty$. Hence the series converges in the open interval $(2,4)$.

Example 2. Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty}\frac{n}{2^n}x^n \tag{12.8}$$

Solution 2. We apply the ratio test. We have

$$\lim_{n\to\infty}\left|\frac{(n+1)x^{(n+1)}}{2^{n+1}}\frac{2^n}{nx^n}\right| = \frac{|x|}{2}\lim_{n\to\infty}\frac{n+1}{n} = \frac{|x|}{2}. \tag{12.9}$$

Thus the series converges absolutely for $|x|<2$, or $-2<x<2$, and diverges for $|x|>2$ or $x>2$ and $x<-2$. At the endpoints $x=2$ and $x=-2$, the series diverges since the nth term does not approach zero as $n\to\infty$. The radius of convergence of the power series is $\rho=2$.

12.3. Series Solutions near an Ordinary Point, Part I.

In previous sections we described methods of solving second order linear differential equations with constant coefficients. We now consider methods of solving second order linear equations when the coefficients are functions of the independent variable. It is sufficient to consider the homogeneous equation

$$P(x)y''+Q(x)y'+R(x)y=0 \tag{12.10}$$

since the procedure for the corresponding nonhomogeneous equation is similar.

For the present, suppose that P, Q and R are polynomials and that they have no common factors. Suppose that we wish to solve Eq. (12.10) in the neighborhood of a point x_0.

A point x_0 such that $P(x_0) \neq 0$ is called an **ordinary point**. Since P is continuous, it follows that there is an interval about x_0 in which $P(x)$ is never zero. In this section we will find series solutions to Eq. (12.10) near an ordinary point x_0.

On the other hand, if $P(x_0) = 0$, then x_0 is called a **singular point** of Eq. (12.10). We look into an example directly.

Example 3. Find a series solution of the equation

$$y'' - xy' - y = 0 \tag{12.11}$$

Solution 3. Here $P(x) = 1, Q(x) = -x$ and $R(x) = -1$. Hence we could pick out ordinary point to be $x_0 = 0$ and find a solution near this point. We look for a solution in the form of a power series about $x_0 = 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n \tag{12.12}$$

and assume that the series converges in some interval $|x| < \rho$. Differentiating Eq. (12.12) term by term yields

$$y' = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1} \tag{12.13}$$

(12.14)

$$y'' = 2a_2 + 2 \cdot 3 a_3 x + \cdots + n(n-1) a_n x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the series (12.13) and (12.14) for y and y'' and y in (12.11) gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \tag{12.15}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

Shifting the index of the term on the left and shifting the first term on the right so that both n starts from zero, we have,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty} na_nx^n + \sum_{n=0}^{\infty} a_nx^n \tag{12.16}$$

Note the goal here is to make the index and the degree of x on all the three summations to be the same. We do this here by making the index n start from zero and the degree of x being n throughout.

Equating the coefficient of x^n from both sides we have,

$$(n+2)(n+1)a_{n+2} = na_n + a_n = (n+1)a_n. \tag{12.17}$$

Simplifying we have our **recurrence relation**,

$$(n+2)a_{n+2} = a_n. \tag{12.18}$$

or

$$a_{n+2} = \frac{a_n}{n+2} \tag{12.19}$$

Since a_{n+2} is given in terms of a_n, the a's are determined in steps of two. Thus a_0 determines a_2, which in turn determines $a_4, \ldots$; a_1 determines a_3 which in turn determines $a_5, \ldots$. For the even numbered coefficients we have

$$a_2 = \frac{a_0}{2}, \quad a_4 = \frac{a_2}{4} = \frac{a_0}{2 \cdot 4}, \quad a_6 = \frac{a_4}{6} = \frac{a_0}{2 \cdot 4 \cdot 6}, \ldots \tag{12.20}$$

These results suggest that in general, if $n = 2k$, then

$$a_n = a_{2k} = \frac{a_0}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2k} = \frac{a_0}{2^k k!}, \quad k = 1, 2, 3, \ldots \tag{12.21}$$

Similarly, for the odd-numbered coefficients we have

$$a_3 = \frac{a_1}{3}, \quad a_5 = \frac{a_3}{5} = \frac{a_1}{3 \cdot 5}, \quad a_7 = \frac{a_5}{7} = \frac{a_1}{3 \cdot 5 \cdot 7}, \ldots \tag{12.22}$$

Similarly these results suggest that in general, if $n = 2k+1$, then

$$a_n = a_{2k+1} = \frac{a_1}{3 \cdot 5 \cdot 7 \cdot \ldots \cdot 2k+1} = \frac{2^k k! a_1}{(2k+1)!} \tag{12.23}$$

Substituting these coefficients into Eq. (12.12), we have

$$\begin{aligned} y &= a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_0}{2^2 2!}x^4 + \frac{2^2 2! a_1}{5!}x^5 + \cdots + \frac{a_0}{2^n n!}x^{2n} + \frac{2^n n! a_1}{(2n+1)!}x^{2n+1} + \cdots \\ &= a_0\left[1 + \frac{1}{2}x^2 + \frac{1}{2^2 2!}x^4 + \cdots + \frac{1}{2^n n!}x^{2n} + \cdots\right] \\ &\quad + a_1\left[x + \frac{1}{3}x^3 + \frac{2^2 2!}{5!}x^5 + \cdots + \frac{2^n n!}{(2n+1)!}x^{2n+1}\right] \end{aligned}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{1}{2^n n!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)!} x^{2n+1}$$

It is easy to see by ratio test that both of these series converge for all x (Check it!).