The Laplace Transform
Lecture 11

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11.1. Outline of Lecture

- Impulse Functions.
- The Convolution Integral.

11.2. Impulse Functions.

In this section we look at functions of impulsive nature, i.e. forces of large magnitude that act over short time intervals. A mechanical interpretation might be the use of a hammer to strike an object or the striking of a baseball with a bat. We would like to have a mathematical way of representing these types of forces.

To do this, we will introduce a new "function", the Dirac delta "function".

11.2.1. The Dirac delta

We define the Dirac delta such that it satisfies the following properties.

Definition 11.1. The Dirac delta at $t = t_0$, denoted by $\delta(t - t_0)$, satisfies the following properties:

(1) $\delta(t - t_0) = 0, \quad t \neq t_0,$

(2) $\displaystyle\int_{t_0-\tau}^{t_0+\tau} \delta(t - t_0)\, dt = 1, \quad \text{for any } \tau > 0,$

(3) $\displaystyle\int_{t_0-\tau}^{t_0+\tau} f(t)\delta(t - t_0)\, dt = f(c), \quad \text{for any } \tau > 0.$

We can think of $\delta(t - t_0)$ as having an "infinite" value at $t = t_0$, so that its total energy is 1, all concentrated at that point. So the Dirac delta can be thought of as an instantaneous impulse at $t = t_0$.

11.2.2. The Laplace transform of the Dirac delta

To solve initial value problems involving the Dirac delta, we need to know its Laplace transform. By the third property of the Dirac delta,

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,dt = e^{-t_0 s}, \quad c > 0. \tag{11.2}$$

also,

(11.3)

$$\mathcal{L}\{f(t)\delta(t-t_0)\} = \int_0^\infty e^{-st}f(t)\delta(t-t_0)\,dt = f(t_0)e^{-t_0 s}, \quad c > 0.$$

We look into an example below,

Example 1. Find the solution of the given initial value problem.

$$y'' - y = -20\delta(t-3), \qquad y(0) = 1, y'(0) = 0. \tag{11.4}$$

Solution 1.

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = -20\mathcal{L}\{\delta(t-3)\}$$
$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - \mathcal{L}\{y\} = -20e^{-3s}$$
$$s^2\mathcal{L}\{y\} - s - \mathcal{L}\{y\} = -20e^{-3s}$$
$$\mathcal{L}\{y\}(s^2-1) = -20e^{-3s} + s.$$
$$\mathcal{L}\{y\} = \frac{-20e^{-3s}}{s^2-1} + \frac{s}{s^2-1}$$
$$y = -20\mathcal{L}^{-1}\{e^{-3s}\frac{1}{s^2-1}\} + \mathcal{L}^{-1}\{\frac{s}{s^2-1}\}$$

By # 8 and # 13 from the table on Page 317, we have

$$y = -20u_3(t)f(t-3) + \cosh t.$$

where $f(t) = \mathcal{L}^{-1}\{\frac{1}{s^2-1}\} = \sinh t$.

Therefore the solution to the initial value problem is

$$y = -20u_3(t)\sinh(t-3) + \cosh t. \tag{11.5}$$

11.3. The Convolution Integral.

If a Laplace transform $H(s)$ can be written as the product of two other transforms $F(s)$ and $G(s)$, then a good question to ask is whether the same is true for their inverse Laplace transform, i.e. whether $\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)\}\mathcal{L}^{-1}\{G(s)\}$. However, this is not the case. In this section we look into exact relation between the inverse Laplace transforms.

Theorem 11.6. *If* $F(s) = \mathcal{L}\{f(t)\}$ *and* $G(s) = \mathcal{L}\{g(t)\}$ *both exist for* $s > a \geq 0$*, then*

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \quad s > a, \tag{11.7}$$

where

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{11.8}$$

The function h *is known as the* **convolution** *of* f *and* g*; the integral in Eq. (11.8) are known as convolution integrals.*

It is convenient to emphasize that the convolution integral can be thought of as a "generalized product" by writing

$$h(t) = (f \star g)(t). \tag{11.9}$$

The convolution $f \star g$ has many of the properties of ordinary multiplication. For example, it is relatively simple to show that

$$f \star g = g \star f \tag{11.10}$$

$$f \star (g_1 + g_2) = f \star g_1 + f \star g_2 \tag{11.11}$$

$$(f \star g) \star h = f \star (g \star h) \tag{11.12}$$

$$f \star 0 = 0 \star f = 0. \tag{11.13}$$

However there are other properties of ordinary multiplication that the convolution integral does not have such as

$$f \star 1 \neq f. \tag{11.14}$$

We look into an example below.

Example 2. Find the inverse Laplace transform of

$$H(s) = \frac{a}{s^2(s^2+s^2)}. \tag{11.15}$$

Solution 2. We can think of

$$H(s) = F(s) \cdot G(s) = \frac{1}{s^2} \cdot \frac{a}{s^2+a^2} \tag{11.16}$$

By # 3 and # 5 of the table on Page 317,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t, \text{ and } g(t) = \mathcal{L}^{-1}\{G(s)\} = \sin at. \tag{11.17}$$

By Theorem (11.6), the inverse transform of $H(s)$ is

$$h(t) = \int_0^t (t-\tau)\sin a\tau\,d\tau. \tag{11.18}$$

Using integration by parts, we have

$$\int_0^t (t-\tau)\sin a\tau \, d\tau = \frac{at - \sin at}{a^2}. \tag{11.19}$$

Therefore

$$\mathcal{L}^{-1}\{H(s)\} = \frac{at - \sin at}{a^2}. \tag{11.20}$$