The Laplace Transform
Lecture 10

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10.1. Outline of Lecture

- Step Functions.
- Differential Equations with Discontinuous Forcing Functions.

10.2. Step Functions.

In this section we look at functions which have jump discontinuities. Differential equations whose right side is a function of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems. We develop some additional properties of Laplace transform in this section and the next which will help us in the solution of such problems.

To deal with functions with jump discontinuities we introduce a function known as the **unit step function** or **Heaviside function**. This function is denoted by u_c and is defined for $c \geq 0$ by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c \end{cases}$$

We want to write a piecewise continuous function in a more "compact" manner with the help of the step function in order to find its Laplace transform. We look into an example below where we do this.

Example 1. Consider the function

$$f(t) = \begin{cases} 2, & 0 \leq t < 2, \\ 5, & 2 \leq t < 5, \\ -3, & 5 \leq t < 9, \\ 3, & t \geq 9 \end{cases}$$

Express $f(t)$ in terms of $u_c(t)$.

Solution 1. We start with the function $f_1(t) = 2$ which agrees with $f(t)$ on $[0, 2)$. To produce the jump of three units (going from 2 to 5) at $t = 2$, we add $3u_2(t)$ to $f_1(t)$, obtaining

$$f_2(t) = 2 + 3u_2(t). \tag{10.1}$$

which agrees with $f(t)$ on $[0, 7)$. The negative jump of eight units (going from 5 to -3) at $t = 5$ corresponds to adding $-8u_5(t)$, which gives

$$f_3(t) = 2 + 3u_2(t) - 8u_5(t). \tag{10.2}$$

Finally to get the positive jump of six units (going from -3 to 3) at $t = 9$, we add $6u_9(t)$. Thus we obtain

$$f(t) = 2 + 3u_2(t) - 8u_5(t) + 6u_9(t). \tag{10.3}$$

The Laplace transform of u_c is easily determined

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st}u_c(t)\,dt = \int_0^c e^{-st}u_c(t)\,dt + \int_c^\infty e^{-st}u_c(t)\,dt \tag{10.4}$$

$$= \int_c^\infty e^{-st}\,dt = \frac{e^{-cs}}{s}, \quad s > 0. \tag{10.5}$$

For a given function f defined for $t \geq 0$, we will often want to consider the related function g defined by

$$y = g(t) = \begin{cases} 0, & t < c, \\ f(t-c), & t \geq c, \end{cases} \tag{10.6}$$

In terms of the unit step function we can write $g(t)$ in the convenient form

$$g(t) = u_c(t)f(t-c). \tag{10.7}$$

We look into the first theorem where we find the Laplace transform of $g(t)$.

Theorem 10.8. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then*

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \quad s > a. \tag{10.9}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}. \tag{10.10}$$

Proof. Check the text book. □

We look into an example which uses this theorem.

Example 2. Find the inverse transform of

$$G(s) = \frac{2e^{-2s}}{s^2-4} \tag{10.11}$$

Solution 2.

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\frac{2e^{-2s}}{s^2-4}\} = \mathcal{L}^{-1}\{e^{-2s}\frac{2}{s^2-4}\} = \mathcal{L}^{-1}\{e^{-2s}F(s)\} \tag{10.12}$$

where $F(s) = \dfrac{2}{s^2-4}$. By the converse of Theorem 10.8,

$$\mathcal{L}^{-1}\{e^{-2s}\frac{2}{s^2-4}\} = u_2(t)f(t-2) \tag{10.13}$$

where $f(t) = \mathcal{L}^{-1}\{\dfrac{2}{s^2-4}\} = \sinh 2t$.

Therefore $f(t-2) = \sinh(2(t-2)) = \sinh(2t-4)$. Hence

$$\mathcal{L}^{-1}\{G(s)\} = u_2(t)\sinh(2t-4). \tag{10.14}$$

We look into another theorem that contains another very useful property of Laplace transform that is somewhat analogous to the previous theorem.

Theorem 10.15. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant, then*

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \qquad s > a+c. \tag{10.16}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{10.17}$$

Proof.

$$\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st}\cdot e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt = F(s-c).$$

□

We look into an example which uses the above theorem.

Example 3. Find the inverse transform of

$$G(s) = \frac{3!}{(s-2)^4} \tag{10.18}$$

Solution 3.

$$G(s) = \frac{3!}{(s-2)^4} = F(s-2) \tag{10.19}$$

where $F(s) = \dfrac{3!}{s^4}$.

$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\dfrac{3!}{s^4}\} = t^3 = f(t)$ by the converse of Theorem 10.15.

By Theorem 10.15,

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{F(s-2)\} = e^{2t}t^3. \tag{10.20}$$

10.3. Differential Equations with Discontinuous Forcing Functions.

In this section we turn our attention to solving differential equations in which the nonhomogeneous term is discontinuous. We look into an example below.

Example 4. Find the solution of the given initial value problem.

$$y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1, \quad f(t) = \begin{cases} 1, & 0 \le t \le 3\pi \\ 0, & 3\pi \le t < \infty \end{cases} \tag{10.21}$$

Solution 4. Using the step function

$$f(t) = 1 - u_{3\pi}(t). \tag{10.22}$$

Therefore the equation becomes

$$y'' + y = 1 - u_{3\pi}(t). \tag{10.23}$$

$$\begin{aligned} \mathcal{L}\{y''\} + \mathcal{L}\{y\} &= \mathcal{L}\{1\} - \mathcal{L}\{u_{3\pi}(t)\} \qquad (10.24)\\ s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} &= \frac{1}{s} - \frac{e^{-3\pi s}}{s} \\ s^2\mathcal{L}\{y\} - 1 + \mathcal{L}\{y\} &= \frac{1}{s} - \frac{e^{-3\pi s}}{s} \\ \mathcal{L}\{y\}(s^2+1) &= \frac{1}{s} - \frac{e^{-3\pi s}}{s} + 1 \\ \mathcal{L}\{y\} &= \frac{1}{s(s^2+1)} - \frac{e^{-3\pi s}}{s(s^2+1)} + \frac{1}{s^2+1} \\ y = \mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} &- \mathcal{L}^{-1}\{\frac{e^{-3\pi s}}{s(s^2+1)}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+1}\} \end{aligned}$$

We use partial fractions(Check Lecture Notes 9) to write

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \tag{10.25}$$

Therfore

$$y = \mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} - \mathcal{L}^{-1}\{e^{-3\pi s}\frac{1}{s}\} - \mathcal{L}^{-1}\{e^{-3\pi s}\frac{s}{s^2+1}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+1}\}$$

By Theorem 10.8,

$$\mathcal{L}^{-1}\{e^{-3\pi s}\frac{1}{s}\} = u_{3\pi}(t)f(t-3\pi). \tag{10.26}$$

and

$$\mathcal{L}^{-1}\{e^{-3\pi s}\frac{s}{s^2+1}\} = u_{3\pi}(t)g(t-3\pi). \tag{10.27}$$

where $f(t) = \mathcal{L}^{-1}\{\frac{1}{s}\} = 1$ and $g(t) = \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = \cos t$. Therefore

$$\mathcal{L}^{-1}\{e^{-3\pi s}\frac{1}{s}\} = u_{3\pi}(t)f(t-3\pi) = u_{3\pi}(t).$$

and

$$\mathcal{L}^{-1}\{e^{-3\pi s}\frac{s}{s^2+1}\} = u_{3\pi}(t)g(t-3\pi) = u_{3\pi}(t)\cos(t-3\pi).$$

Therefore the solution to the differential equation is

$$y = 1 - \cos t - u_{3\pi}(t) - u_{3\pi}(t)\cos(t-3\pi) + \sin t. \tag{10.28}$$

Since $\cos(t-3\pi) = \cos t$, therefore

$$y = 1 - \cos t - u_{3\pi}(t) - u_{3\pi}(t)\cos t + \sin t. \tag{10.29}$$