An Introduction to Ordinary Differential Equations Lecture 1

Dibyajyoti Deb

1.1. Outline of Lecture

- What is a Differential Equation?
- Solutions of Some Differential Equations
- Classifying Diff. Eqns.: Order, Linear vs. Nonlinear

1.2. What is a Differential Equation?

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms, the relations are equations and the rates are derivatives. Equations containing derivatives are **differential equations**. You've probably all seen an ordinary differential equation (ODE); for example the physical law that governs the motion of objects is Newton's second law, which states that the mass of the object times its acceleration is equal to the net force on the object. In mathematical terms this law is expressed by the equation

$$F = m\frac{dv}{dt}. \tag{1.1}$$

where m is the mass of the object, v is its velocity, t is the time and F is the net force exerted on the object. Here t is the **independent variable** and v is the **dependent variable**. This is an ODE because there is only one independent variable, here t which represents time.

A partial differential equation (PDE) relates the partial derivatives of a function of two or more independent variables together. For example, the heat conduction equation,

$$\alpha^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \tag{1.2}$$

arises in many places in mathematics and physics. Here α^2 is a physical constant. For simplicity, we can use subscript notation for partial derivatives, so this equation can also be written $\alpha^2 u_{xx} = u_t$.

We say a function is a **solution** to a differential equation if it satisfies the equation and any side conditions given. Mathematicians are often interested in if a solution **exists** and when it is **unique**.

1.3. Solution to some differential equations

Let's look into the differential equation from the previous section in more detail. What are the forces that act on the object as it falls?

Gravity exerts a force equal to the weight of the object, or mg, where g is the acceleration due to gravity. There is also a force due to air resistance, or drag, that is more difficult to model. Let's assume that the drag force is proportional to the velocity. Thus the drag force has a magnitude γv, where γ is a constant called the drag coefficient.

Gravity always acts in the downward (positive) direction, whereas drag acts in the upward (negative) direction. Thus

$$m\frac{dv}{dt} = mg - \gamma v \tag{1.3}$$

To solve this equation, divide by m. Note that m, g and γ are constants for this model.

$$\frac{dv}{dt} = g - \frac{\gamma}{m}v \tag{1.4}$$

We would like to isolate the terms involving v and t such that we can integrate both sides.

$$\frac{dv}{dt} = -\frac{\gamma}{m}(v - \frac{gm}{\gamma}) \tag{1.5}$$

After cross multiplying

$$\frac{dv}{v - \frac{gm}{\gamma}} = -\frac{\gamma}{m}dt \tag{1.6}$$

Integrate both sides

$$\int \frac{dv}{v - \frac{gm}{\gamma}} = \int -\frac{\gamma}{m}dt$$

$$\ln|v - \frac{gm}{\gamma}| = -\frac{\gamma t}{m} + C$$

$$v - \frac{gm}{\gamma} = \pm e^{-\frac{\gamma t}{m}+C}$$

Simplifying the right side and replacing $\pm e^C$ with C_1 (another constant), we have,

$$v(t) = \frac{gm}{\gamma} + C_1 e^{-\frac{\gamma t}{m}} \tag{1.7}$$

This expression contains all possible solutions of Eq. (1.4) and is called the **general solution**. The geometrical representation of the general solution 1.7 is an infinite family of curves called **integral curves**. Each integral curve is associated with a particular value of C_1 and is the graph corresponding to that value of C_1.

Now if the ball is "dropped" from a certain height then it is clear that the initial velocity is zero. Therefore $v(0) = 0$. We can use this additional condition to determine C_1. This is an example of an **initial condition**. The differential equation together with the initial condition form an **initial value problem**.

1.4. Classifying Diff. Eqns.: Order, Linear vs. Nonlinear

We have already discussed the difference between an ordinary and a partial differential equation. When classifying differential equations we need to look at the *order* of a differential equation. The **order** of a differential equation is the order of the highest derivative present in the equation. For example Eq. 1.4 is a first order ODE. More generally, the equation

$$F[t, x(t), x'(t), \ldots, x^{(n)}(t)] = 0 \tag{1.8}$$

is an ordinary differential equation of the nth order.

A crucial classification of differential equations is whether they are linear or nonlinear. The ordinary differential equation

$$f(t, y, y', \ldots, y^{(n)}) = 0, \tag{1.9}$$

is said to be **linear** if F is a linear function of the variable $y, y', \ldots, y^{(n)}$; a similar definition applies to partial differential equations. Thus the general linear ordinary differential equation of order n is

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_0(t)y = g(t). \tag{1.10}$$

The most important thing to note about linear differential equations is that there are no products of the function, $y(t)$, and its derivatives and neither the function or its derivatives are used in determining if a differential equation is linear. For example Eq. 1.4 is a linear equation.

An equation that is not of the form 1.10 is a **nonlinear** equation. An example of a nonlinear equation would be

$$y'' + 2yy' = t^3 \tag{1.11}$$

because of the presence of the term yy'. We will be looking at differences between linear and nonlinear equations in more detail in a later lecture.